

# LIMITING DISTRIBUTION OF EIGENVALUES IN THE LARGE SIEVE MATRIX

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**ABSTRACT.** The large sieve inequality is equivalent to the bound  $\lambda_1 \leq N + Q^2 - 1$  for the largest eigenvalue  $\lambda_1$  of the  $N$  by  $N$  matrix  $A^*A$ , naturally associated to the positive definite quadratic form arising in the inequality. For arithmetic applications the most interesting range is  $N \asymp Q^2$ . Based on his numerical data Ramaré conjectured that when  $N \sim \alpha Q^2$  as  $Q \rightarrow \infty$  for some finite positive constant  $\alpha$ , the limiting distribution of the eigenvalues of  $A^*A$ , scaled by  $1/N$ , exists and is non-degenerate. In this paper we prove this conjecture by establishing the convergence of all moments of the eigenvalues of  $A^*A$  as  $Q \rightarrow \infty$ . Previously only the second moment was known, due to Ramaré. Furthermore, we obtain an explicit description of the moments of the limiting distribution, and establish that they vary continuously with  $\alpha$ . Some of the main ingredients in our proof include the large-sieve inequality and results on  $n$ -correlations of Farey fractions.

## 1. INTRODUCTION

Let  $\mathcal{F}_Q$  denote the set of Farey fractions of order  $Q$ , that is the set of reduced fractions  $\frac{a}{q}$  with  $0 < a \leq q \leq Q$ . In particular  $|\mathcal{F}_Q| = \sum_{q \leq Q} \varphi(q) \sim \frac{3}{\pi^2} Q^2$  as  $Q \rightarrow \infty$ . The large sieve inequality states that, for any sequence of complex numbers  $a(n)$ ,

$$\sum_{\theta \in \mathcal{F}_Q} \left| \sum_{n \leq N} a(n) e(n\theta) \right|^2 \leq (N + Q^2 - 1) \sum_{n \leq N} |a(n)|^2. \quad (1.1)$$

The large sieve was first discovered by Linnik [11], who applied it to bound the number of moduli  $q$  for which the least quadratic non-residue exceeds  $q^\epsilon$ . Since its inception the large sieve fascinated analytic number theorists, not the least because of the variety of its incarnations (probabilistic [16], arithmetic [11], analytic [4]). The form (1.1) is the outcome of a long chain of improvements, due among others to Bombieri [5], Bombieri-Davenport [4], Gallagher [9], Montgomery [13], Montgomery-Vaughan [12], Rényi [16], Roth [17], Selberg [18], ... One of the major applications of (1.1) is the Bombieri-Vinogradov [3] theorem on primes in arithmetic progressions.

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A fruitful point of view is to interpret (1.1) in terms of the eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$  of the  $N \times N$  symmetric positive definite matrix,

$$A^*A = \left( \sum_{\theta \in \mathcal{F}_Q} e((n_1 - n_2)\theta) \right)_{1 \leq n_1, n_2 \leq N} \text{ where } A = \left( e(n\theta) \right)_{\substack{\theta \in \mathcal{F}_Q \\ 1 \leq n \leq N}}.$$

Note that  $\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \dots \geq \sqrt{\lambda_N} \geq 0$  are the singular values of  $A$  and the following identity holds trivially:

$$\sum_{i \leq N} \lambda_i = \text{Tr}(A^*A) = |\mathcal{F}_Q|N.$$

Since  $\|A\mathbf{v}\|^2$  is equal to (1.1) when  $\mathbf{v} = (a(1), \dots, a(n))$  and  $\lambda_1 = \|A\|^2 = \|A^*A\| = \|AA^*\|$  the large sieve inequality (1.1) is equivalent to  $\lambda_1 \leq N + Q^2 - 1$ . It is very desirable, from the point of view of applications, to replace the inequality (1.1) by an asymptotic equality. In the range  $N < Q^{2-\varepsilon}$  one can adapt the results of Conrey-Iwaniec-Soundararajan [6] to obtain an asymptotic for a class of sequences  $a(n)$ .

We would like to investigate the problem of refining the large sieve inequality to an asymptotic equality in wide generality, and in particular in the range  $N \asymp Q^2$ . This range is particularly interesting from an arithmetic point of view; for example it comes up naturally in the proof of the explicit Brun-Titchmarsh theorem. As a first step in this direction, one would like to understand the limiting distribution of the eigenvalues of  $A^*A$ , that is the limiting distribution of the sequence of probability measures on  $[0, \infty)$  given by

$$\mu_{Q,N} = \frac{1}{N} \sum_{i \leq N} \delta_{\lambda_i/N},$$

where  $\delta_\lambda$  denotes the Dirac probability measure supported at  $\lambda \in \mathbb{R}$ . It turns out that this is relatively easy when the ratio  $Q^2/N$  either tends to infinity or to zero. When  $N/Q^2 \rightarrow \infty$  as  $Q \rightarrow \infty$ , then since the rank of  $A^*A$  is  $\leq Q^2$ , it follows that most eigenvalues are zero, therefore  $\mu_{Q,N} \rightarrow \delta_0$ . On the other hand, when  $N/Q^2 \rightarrow 0$  as  $Q \rightarrow \infty$ , then according to a deeper result of Ramaré [14] concerning the asymptotic behaviour of  $\sum_{i \leq N} \lambda_i^2$ , one concludes that when  $N/Q^2 \rightarrow 0$  all but  $o(N)$  of the eigenvalues cluster close to  $|\mathcal{F}_Q|$ . We will be concerned with the remaining regime  $N \asymp Q^2$ .

In [14, 15] Ramaré conducted several numerical experiments that suggested the existence of a *non-degenerate* limiting distribution function as soon as the ratio  $\frac{N}{Q^2}$  tends to a finite limit with  $Q \rightarrow \infty$ . In support of the numerical data Ramaré established in [14] the convergence of the second moment

$$\mathfrak{M}_Q(2) := \int_0^\infty t^2 d\mu_{Q,N}(t) = \frac{1}{N} \sum_{i \leq N} \left( \frac{\lambda_i}{N} \right)^2$$

as  $Q \rightarrow \infty$ . The form of the second moment (in particular its variation with  $\frac{N}{Q^2}$ ) ruled out the possibility of convergence to any standard probability law.

In this paper we estimate all moments of  $\mu_{Q,N}$ ,

$$\begin{aligned} \mathfrak{M}_Q(\ell) &:= \int_0^\infty t^\ell d\mu_{Q,N}(t) = \frac{1}{N} \sum_{i \leq N} \left( \frac{\lambda_i}{N} \right)^\ell = \frac{1}{N} \text{Tr} \left( \frac{A^* A}{N} \right)^\ell \\ &= \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ 1 \leq n_1, \dots, n_\ell \leq N}} e((n_1 - n_2)\theta_1 + (n_2 - n_3)\theta_2 + \dots + (n_\ell - n_1)\theta_\ell), \end{aligned} \quad (1.2)$$

and prove Ramaré's conjecture, that is the weak\*-convergence of  $\mu_{Q,N}$  to a limiting distribution when  $N \sim \alpha Q^2$ :

**Corollary 1.** *Suppose that  $N \sim \alpha Q^2$  as  $Q \rightarrow \infty$  for some fixed constant  $\alpha \in (0, \infty)$ . There exists a non-degenerate probability measure  $\mu_\alpha$  on  $[0, \infty)$  such that*

$$\mu_{Q,N} \xrightarrow{w^*} \mu_\alpha$$

as  $Q \rightarrow \infty$ . Moreover  $\mu_\alpha$  is determined by its moments

$$M_\ell(\alpha) = \int_0^\infty t^\ell d\mu_\alpha(t),$$

explicitly described in Theorem 1 below.

**Remark 2.** In principle our proof delivers a rate of convergence. For example, when  $N = |\mathcal{F}_Q|$  our approach shows that convergence of the  $\ell^{\text{th}}$  moment occurs at a rate of at least  $\ll Q^{-\delta_\ell}$  for some exponent  $\delta_\ell > 0$ . Since our proof reveals that  $\delta_\ell \ll \ell^A$  for some absolute constant  $A > 0$ , it is possible to show by Fourier analytic techniques that when  $N = |\mathcal{F}_Q|$  (and thus  $\alpha = 3/\pi^2$ ) there exists a  $\delta > 0$ , such that for any fixed smooth function  $f$ ,

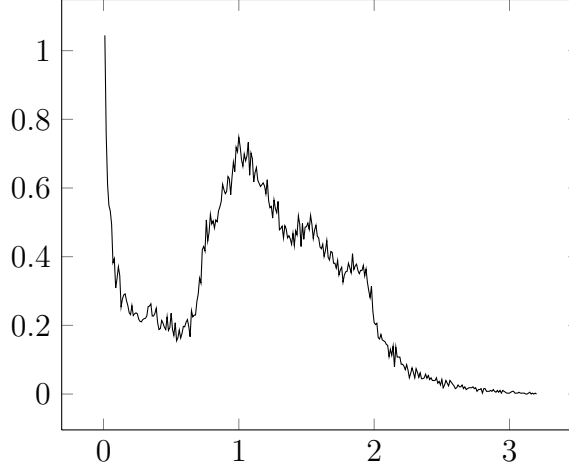
$$\int_0^\infty f(t) d\mu_{Q,N}(t) = \int_0^\infty f(t) d\mu_\alpha(t) + O((\log Q)^{-\delta})$$

as  $Q \rightarrow \infty$ . More generally this holds whenever  $N = \alpha Q^2 + O(Q^{2-\eta})$  for some  $\eta > 0$ .

We include below an empirical approximation for the probability density function of  $\mu_{3/\pi^2}$ , based on an approximation with  $Q = 500$  and  $N = |\mathcal{F}_{500}| = 76116$ <sup>1</sup>. This resembles the previous data obtained by Ramaré [15]. There is a large number of eigenvalues in  $[0, 0.01]$  (roughly 10%) and we have omitted them from the graph (they contribute a disproportionate 6 on the scale of the graph). Note that when  $N = |\mathcal{F}_Q|$  there are no eigenvalues that are equal to 0, since in this case  $\det A$  is just a non-zero Vandermonde determinant.

<sup>1</sup>The computation took less than a day, requiring 8 cores and 44GB of RAM memory. We used a custom C program invoking LAPACK linear algebra routines. The code is available on request.

### Approximation for the p.d.f of $\mu_{3/\pi^2}$



The description of the moments  $M_\ell(\alpha)$  is rather complicated, so we start with some preliminary remarks. Since  $\lambda_1 \leq 1 + \frac{Q^2}{N} \leq 1 + \alpha^{-1} + o(1)$ , all probability measures  $\mu_{Q,N}$  are supported in  $[0, C(\alpha)]$  for some constant  $C(\alpha) > 0$ , so we have for free that

$$0 \leq \mathfrak{M}_Q(\ell) = \int_0^\infty t^\ell d\mu_{Q,N}(t) \leq C(\alpha)^\ell.$$

In particular, if each  $\mathfrak{M}_Q(\ell)$  converges to a limit  $C_\ell$  as  $Q \rightarrow \infty$ , then  $\mu_{Q,N} \xrightarrow{w^*} \mu$  for some probability measure  $\mu$  supported in  $[0, 1 + \alpha^{-1}]$  with moments  $C_\ell$ . We also notice that the first moment is trivial:

$$\mathfrak{M}_Q(1) = \frac{1}{N} \operatorname{Tr} \left( \frac{A^* A}{N} \right) = \frac{|\mathcal{F}_Q|}{N} \sim \frac{3}{\pi^2 \alpha}.$$

For  $\ell \geq 2$ , our starting point will be the analysis of the exponential sum in (1.2). This is a sum of  $\asymp N^{2\ell}$  oscillating terms, which by the large sieve are bounded by  $\ll N^{\ell+1}$ , so close to square-root cancellation. Our task is to refine this bound to an asymptotic equality. We accomplish this in the theorem below.

**Theorem 1.** (i) *For each  $\ell \geq 2$ , there exists a continuous function  $M_\ell(\alpha)$  on  $(0, \infty)$  and some explicit exponent  $\theta_\ell > 0$  such that, given  $0 < \gamma_1 < \gamma_2$ , one has*

$$\mathfrak{M}_Q(\ell) = M_\ell\left(\frac{N}{Q^2}\right) + O_{\ell, \gamma_1, \gamma_2}(Q^{-\theta_\ell})$$

whenever  $\gamma_1 Q^2 \leq N \leq \gamma_2 Q^2$  as  $Q \rightarrow \infty$ . Precisely, taking  $(A, 0) = |A|$ ,  $\mathbf{A} = (A_1, \dots, A_{\ell-1})$ ,  $\mathbf{B} = (B_1, \dots, B_{\ell-1})$ ,  $\text{sinc}(x) := \frac{\sin x}{x}$  if  $x \neq 0$ ,  $\text{sinc}(0) = 1$ , and

$$h_{A,B}(x, y) := \frac{B}{y(Ay - Bx)},$$

$$\mathcal{D}_{\mathbf{A}, \mathbf{B}} := \{(x, y) \in [0, 1]^2 : x \leq y, 0 < A_i y - B_i x \leq 1, \forall i \in [1, \ell - 1]\}, \quad (1.3)$$

we have

$$\begin{aligned} M_\ell(\alpha) &= \frac{6}{\pi^2 \alpha} \sum_{\substack{\mathbf{A}, \mathbf{B} \in \mathbb{Z}^{\ell-1} \\ (A_i, B_i) = 1, \forall i \\ A_i^2 + B_i^2 \neq 0, \forall i}} \iint_{\mathcal{D}_{\mathbf{A}, \mathbf{B}}} \text{sinc}(\pi \alpha h_{A_1, B_1}(x, y)) \text{sinc}(\pi \alpha h_{A_{\ell-1}, B_{\ell-1}}(x, y)) \\ &\quad \times \prod_{i=1}^{\ell-2} \text{sinc}(\pi \alpha h_{A_i, B_i}(x, y) - \pi \alpha h_{A_{i+1}, B_{i+1}}(x, y)) dx dy \in [0, \infty). \end{aligned} \quad (1.4)$$

- (ii) The expression defining  $M_\ell(\alpha)$  above is absolutely convergent.
- (iii) For each  $\ell \geq 2$ , there exists  $\kappa_\ell > 0$  such that, given  $0 < \gamma_1 < \gamma_2$ , one has

$$|M_\ell(\alpha) - M_\ell(\beta)| \ll_{\gamma_1, \gamma_2} |\alpha - \beta|^{\kappa_\ell}$$

for all  $\alpha, \beta \in [\gamma_1, \gamma_2]$ .

We highlight that both continuity and absolute convergence of the expression defining  $M_\ell(\alpha)$  are non-trivial. Parts (ii) and (iii) in Theorem 1 will be proved in Section 4, while part (i) will be proved in Section 5.

With Theorem 1 at hand the deduction of our main result Corollary 1 is immediate.

*Deduction of Corollary 1.* The large sieve inequality yields  $\text{supp } \mu_{Q,N} \subseteq [0, 1 + \frac{Q^2}{N}]$  for all  $Q$ . Hence there exists a positive constant  $K$  such that  $\text{supp } \mu_{Q,N} \subseteq [0, K]$  for all  $Q$ . Banach-Alaoglu's theorem shows that the sequence of probability measures  $(\mu_{Q,N})$  on the compact set  $[0, K]$  has at least one cluster point  $\mu_\alpha$  in the weak\* topology on  $C([0, K])^*$ , and  $\mu_\alpha$  is a probability measure on  $[0, K]$ . Theorem 1 shows that any two such cluster points  $\mu, \mu_\alpha$  have the same moments  $M_\ell(\alpha)$ ,  $\ell \in \mathbb{N}$ , thus  $\mu = \mu_\alpha$  by Stone-Weierstrass.

It remains to show that the limiting distribution  $\mu_\alpha$  is not degenerate, that is its variance  $M_\ell(2) - M_\ell(1)^2 = M_\ell(2) - (\frac{3}{\pi^2 \alpha})^2$  is non-zero. For  $\alpha \geq 1$  this is impossible, since by Proposition 4 from Section 2, the second moment is  $> \frac{3}{\pi^2 \alpha} \geq (\frac{3}{\pi^2 \alpha})^2$ . For  $\alpha < 1$  according to Ramaré's formula or the proof of Proposition 4 below,

$$\int_0^\infty t^2 d\mu_\alpha(t) = \frac{3}{\pi^2 \alpha} + \frac{6}{\pi^2 \alpha} \cdot \frac{1}{\pi i} \int_{-\frac{1}{8}+i\infty}^{-\frac{1}{8}+i\infty} \frac{\alpha^{s-1} \zeta(s)}{s(s+1)(2-s)^2 \zeta(2-s)} ds.$$

Shifting the contour of integration to  $\Re s = 1$ , we collect a pole at  $s = 0$  that contributes  $(\frac{3}{\pi^2\alpha})^2$ . We conclude that

$$\left| \int_0^\infty t^2 d\mu_\alpha(t) - \left( \frac{3}{\pi^2\alpha} \right)^2 \right| > \frac{3}{\pi^2\alpha} - \left| \frac{6}{\pi^2\alpha} \cdot \frac{1}{\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\alpha^{s-1} \zeta(s)}{s(s+1)(2-s)^2 \zeta(2-s)} ds \right|. \quad (1.5)$$

Notice that the rightmost term is **strictly** less than

$$\frac{6}{\pi^2\alpha} \cdot \frac{1}{\pi} \int_{\mathbb{R}} \frac{dt}{|1+it|^4} = \frac{3}{\pi^2\alpha}.$$

It follows that the left-hand side of (1.5) is  $> 0$  and hence the distribution  $\mu_\alpha(t)$  is not degenerate.  $\square$

One wonders if  $\mu_\alpha$  is absolutely continuous with respect to the Lebesgue measure, except for possible atoms at 0 (which arise naturally when  $N > (1+\varepsilon)|\mathcal{F}_Q|$ , since  $A$  is not of full rank as soon as  $N > |\mathcal{F}_Q|$ ).

The result in [2] shows (after a small modification) that when  $N = |\mathcal{F}_Q|$ , there exists a positive measure  $g_\ell$ , supported on  $[\frac{3}{\pi^2}, \infty)$  when  $\ell = 2$  and on a countable union of surfaces in  $\mathbb{R}^{\ell-1}$  when  $\ell > 2$  (thus in particular having Lebesgue measure zero support when  $\ell > 3$ ), such that,

$$S_\ell(Q; f) := \frac{1}{N} \sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ \text{distinct}}} F_Q(\theta_1 - \theta_2, \theta_2 - \theta_3, \dots, \theta_{\ell-1} - \theta_\ell) = 2 \int_{[0, \infty)^{\ell-1}} f dg_\ell + o(1),$$

as  $Q \rightarrow \infty$ , for smooth functions  $f$  compactly supported in  $(0, \infty)^{\ell-1}$ , where  $F_Q$  denotes the  $\mathbb{Z}^{\ell-1}$ -periodization of  $f$  given by

$$F_Q(\mathbf{y}) := \sum_{\mathbf{m} \in \mathbb{Z}^{\ell-1}} f(N(\mathbf{m} + \mathbf{y})), \quad \mathbf{y} \in \mathbb{R}^{\ell-1} / \mathbb{Z}^{\ell-1}.$$

Concretely, the measure  $g_\ell$  is supported on the union of all surfaces  $\Phi_{\mathbf{A}, \mathbf{B}}(\mathcal{D}_{\mathbf{A}, \mathbf{B}})$  with  $\mathbf{A}, \mathbf{B} \in \mathbb{N}^{\ell-1}$ ,  $(A_i, B_i) = 1, \forall i$ ,  $\Phi_{\mathbf{A}, \mathbf{B}} = T \circ T_{\mathbf{A}, \mathbf{B}}$ ,  $T_{\mathbf{A}, \mathbf{B}} = \frac{3}{\pi^2}(h_{A_1, B_1}, \dots, h_{A_{\ell-1}, B_{\ell-1}})$  and  $T(x_1, \dots, x_{\ell-1}) = (x_1 - x_2, x_2 - x_3, \dots, x_{\ell-2} - x_{\ell-1}, x_{\ell-1})$ . It is important here that the support of  $f$  is compact, as it implies that the number of  $(2\ell - 2)$ -tuples  $(\mathbf{A}, \mathbf{B})$  that produce non-zero terms in  $\int_{[0, \infty)^{\ell-1}} f dg_\ell$  is finite. However, when the support of  $f$  contains 0 or when  $f$  is not compactly supported, the question of convergence of such an expression becomes delicate, in particular since we do not have non-trivial point-wise bounds for  $g_\ell$  as soon as  $\ell > 2$ .

There is a close relationship between (1.4) and the density  $g_\ell$ . Using the absolute convergence of (1.4) and an explicit formula for  $g_\ell$ , provided by instance by formula

(1.4) in [2], it is possible to re-write  $M_\ell(\alpha)$  in (1.4) as

$$\frac{6}{\pi^2 \alpha} \int_{[0, \infty)^{\ell-1}} \text{sinc} \left( \frac{\pi^3 \alpha (x_1 + \dots + x_{\ell-1})}{3} \right) \prod_{i=1}^{\ell-1} \text{sinc} \left( \frac{\pi^3 \alpha x_i}{3} \right) d\tilde{g}_\ell(x_1, \dots, x_{\ell-1}), \quad (1.6)$$

where the measure  $\tilde{g}_\ell$  is defined in a similar way as  $g_\ell$ , but summing over the larger range  $(\mathbf{A}, \mathbf{B}) \in \mathbb{Z}^{2\ell-2}$  with  $(A_i, B_i) = 1$  and  $A_i^2 + B_i^2 \neq 0$  for all  $i$  (in particular the support of  $\tilde{g}_\ell$  still has zero Lebesgue measure in  $\mathbb{R}^{\ell-1}$ ). From this we see that the proof of absolute convergence of the expression defining  $M_\ell(\alpha)$  amounts to establishing bounds for the decay rate of  $\tilde{g}_\ell$  in an averaged sense.

We close by mentioning that two of the remaining challenges are to determine finer properties of the distribution function of the limiting probability measure  $\mu_\alpha$  and to obtain information about the limiting eigenvectors of the large sieve matrix  $A^*A$ . We hope to come back to these questions in a later paper.

**1.1. Outline of the argument and plan of the paper.** We now highlight the main steps in our proof. We first address in Section 2 the case  $\ell = 2$ , adapting techniques from [2]. This recovers Ramaré's initial result. It is not clear how to proceed when  $\ell \geq 3$  without introducing a smoothing on the  $n_1, \dots, n_\ell$  variables. Since our sum is highly oscillating, it is also not immediately clear that a smoothing can be efficiently introduced. Ramaré remarks in his paper [14] that this is a significant stumbling block. In Section 3 we show that one can introduce a substantial smoothing by using the large sieve inequality. After having smoothed, we would like to relate the question of computing the moments to the  $n$ -correlation function of Farey fractions which was computed in [2]. Here an initial obstacle is that the variables  $\theta_i$  are chained in a circular manner, requiring us to control simultaneously  $N(\theta_1 - \theta_2), N(\theta_2 - \theta_3), \dots, N(\theta_\ell - \theta_1)$ . We resolve this problem by using a Fourier analytic trick, which reduces us to the case where we need to understand  $N(\theta_1 - \theta_2), \dots, N(\theta_{\ell-1} - \theta_\ell)$ , that is, without the circular chaining. We then adapt in Section 5 the argument from a paper by Zaharescu and the first author [2] where the higher correlation measures of Farey fractions are computed. One of the key arguments from [2] relies on the divisor switching technique. It is interesting to notice that this is also the crucial ingredient in the recent work on the “asymptotic large sieve” by Conrey-Iwaniec-Soundararajan [6]. Finally, in order to conclude the computation carried out in Section 5, we need to establish the absolute convergence of the expression defining  $M_\ell(\alpha)$ . This requires a rather substantial elementary argument that splits into several cases. We perform this analysis in Section 4. The main ingredient is a counting lemma for simultaneous solutions to a system of equations of the form  $A_i B_{i+1} - A_{i+1} B_i = \Delta_i$ . In a subsequent paper we hope to apply this argument to analyze the behavior at infinity of higher correlation functions of Farey fractions, which was only worked out for the pair correlation.

## 2. MOMENTS OF SECOND ORDER

We first consider in detail the case  $\ell = 2$ . An asymptotic formula for  $\mathfrak{M}_Q(2)$  was previously established in [14]. Here we follow a different approach in the spirit of [2].

Denote by  $H_N$  the characteristic function of the interval  $[\frac{1}{N}, 1]$ . We can write

$$\begin{aligned} \mathfrak{M}_Q(2) &= \frac{1}{N^3} \sum_{\theta_1, \theta_2 \in \mathcal{F}_Q} \sum_{n_1, n_2 \in \mathbb{Z}} e((n_1 - n_2)(\theta_1 - \theta_2)) H_N\left(\frac{n_1}{N}\right) H_N\left(\frac{n_2}{N}\right) \\ &= \frac{1}{N^3} \sum_{\theta_1, \theta_2 \in \mathcal{F}_Q} \sum_{n \in \mathbb{Z}} e(n(\theta_1 - \theta_2)) \sum_{n_2 \in \mathbb{Z}} H_N\left(\frac{n_2}{N}\right) H_N\left(\frac{n_2 + n}{N}\right). \end{aligned} \quad (2.1)$$

The inner sum in (2.1) is seen to coincide with  $(N - |n|)\chi_{[-N, N]}(n)$ , and so

$$\mathfrak{M}_Q(2) = \frac{1}{N^2} \sum_{n=-N}^N \phi\left(\frac{n}{N}\right) \sum_{\theta_1, \theta_2 \in \mathcal{F}_Q} e(n(\theta_1 - \theta_2)), \quad (2.2)$$

where

$$\phi(x) := (1 - |x|)\chi_{[-1, 1]}(x) = (\chi_{[0, 1]} * \chi_{[-1, 0]})(x).$$

Using also

$$\widehat{\chi}_{[0, 1]}(x) = e^{-\pi i x} \operatorname{sinc}(\pi x),$$

we find

$$\psi(x) := \widehat{\phi}(x) = \widehat{\chi}_{[0, 1]}(x) \widehat{\chi}_{[-1, 0]}(x) = \operatorname{sinc}^2(\pi x) = \psi(-x). \quad (2.3)$$

From (2.2) and (2.3) we infer

$$\mathfrak{M}_Q(2) = \frac{1}{N} \sum_{n \in \mathbb{Z}} \frac{1}{N} \widehat{\psi}\left(\frac{n}{N}\right) \sum_{\theta_1, \theta_2 \in \mathcal{F}_Q} e(n(\theta_1 - \theta_2)), \quad (2.4)$$

thus it suffices to reprove an analogue of [2, Theorem 2] with the compactly supported smooth function  $H$  there being replaced by  $\psi$  here.

**2.1. An asymptotic formula for  $\mathfrak{M}_Q(2)$ .** We follow closely Sections 2 and 4 in [2] with

$$c_n = \frac{1}{N} \widehat{\psi}\left(\frac{n}{N}\right).$$

Consider the Möbius function  $\mu$  and the summation function

$$M(X) := \sum_{n \leq X} \mu(n).$$



An application of Möbius inversion shows that (see, e.g., formula (1) in Section 12.2 of [7]), for every function  $f : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{C}$ ,

$$\sum_{\theta \in \mathcal{F}_Q} f(\theta) = \sum_{k \geq 1} M\left(\frac{Q}{k}\right) \sum_{j=1}^k f\left(\frac{j}{k}\right).$$

In particular this provides the following well-known identity:

$$\sum_{\theta \in \mathcal{F}_Q} e(n\theta) = \sum_{d|n} dM\left(\frac{Q}{d}\right), \quad n \in \mathbb{Z}, Q \in \mathbb{N}. \quad (2.5)$$

For  $n = 0$  this corresponds to  $|\mathcal{F}_Q| = \sum_{d \geq 1} M\left(\frac{Q}{d}\right)$ .

Poisson's summation formula [8, Theorem 8.36] holds true when applied to a pair  $(\psi_h, \widehat{\psi}_h)$ , where  $\psi_h(x) := \psi(hx)$ ,  $h > 0$ , because  $|\psi(x)| \leq \frac{1}{(1+|x|)^2}$  and  $\widehat{\psi}$  has compact support. Proceeding exactly as in [2], we arrive at the following closed form analogue of formulas (4.4) and (4.5) in [2]:

$$\begin{aligned} \mathfrak{M}_Q(2) &= \frac{1}{N} \sum_{r_1, r_2 \in [1, Q]} \mu(r_1) \mu(r_2) \sum_{\substack{d_1 \in [1, \frac{Q}{r_1}] \\ d_2 \in [1, \frac{Q}{r_2}]}} (d_1, d_2) \sum_{n \in \mathbb{Z}} \psi\left(\frac{nN}{[d_1, d_2]}\right) \\ &= \frac{1}{N} \sum_{r_1, r_2 \in [1, Q]} \mu(r_1) \mu(r_2) \sum_{\delta \in [1, \min\{\frac{Q}{r_1}, \frac{Q}{r_2}\}]} \delta \sum_{n \in \mathbb{Z}} \sum_{\substack{q_1 \in [1, \frac{Q}{r_1 \delta}] \\ q_2 \in [1, \frac{Q}{r_2 \delta}] \\ (q_1, q_2) = 1}} \psi\left(\frac{nN}{q_1 q_2 \delta}\right). \end{aligned} \quad (2.6)$$

This sum is split as  $\mathfrak{M}_Q^I(2) + \mathfrak{M}_Q^{II}(2) + \mathfrak{M}_Q^{III}(2)$ , with terms arising from the contribution of  $n = 0$ ,  $\psi_\Lambda(x) := \psi(x)\chi_{\{0 < |x| \leq \Lambda\}}$ , and respectively  $\psi(x)\chi_{\{|x| > \Lambda\}}$ , where we take  $\Lambda := N^{1/2} \asymp Q$ .

The contribution of  $n = 0$  is given by

$$\begin{aligned} \mathfrak{M}_Q^I(2) &= \frac{1}{N} \sum_{r_1, r_2 \in [1, Q]} \mu(r_1) \mu(r_2) \sum_{\substack{d_1 \in [1, \frac{Q}{r_1}] \\ d_2 \in [1, \frac{Q}{r_2}]}} (d_1, d_2) \\ &= \frac{1}{N} \sum_{d_1, d_2 \in [1, Q]} (d_1, d_2) M\left(\frac{Q}{d_1}\right) M\left(\frac{Q}{d_2}\right) = \frac{|\mathcal{F}_Q|}{N}, \end{aligned} \quad (2.7)$$

the last equality being noticed at the top of page 420 in [14].

The contribution of  $n$  with  $\frac{|n|N}{q_1 q_2 \delta} > \Lambda$  to the inner sum in (2.6) is

$$\ll \frac{q_1^2 q_2^2 \delta^2}{N^2} \sum_{n > \frac{\Lambda q_1 q_2 \delta}{N}} \frac{1}{n^2} \ll \frac{q_1^2 q_2^2 \delta^2}{N^2} \cdot \frac{N}{\Lambda q_1 q_2 \delta} = \frac{q_1 q_2 \delta}{\Lambda N},$$

hence

$$\mathfrak{M}_Q^{III}(2) \ll \frac{1}{\Lambda N^2} \sum_{r_1, r_2 \in [1, Q]} \sum_{\delta \in [1, \frac{Q}{\max\{r_1, r_2\}}]} \delta^2 \left( \frac{Q}{r_1 \delta} \right)^2 \left( \frac{Q}{r_2 \delta} \right)^2 \ll \frac{Q^4}{\Lambda N^2} \ll Q^{-1}. \quad (2.8)$$

Finally we estimate  $\mathfrak{M}_Q^{II}(2)$ . In this situation we have  $0 < \frac{|n|N}{q_1 q_2 \delta} \leq \Lambda$ , leading to  $N \leq \Lambda q_1 q_2 \delta \leq \Lambda Q \min\{q_1, q_2\}$  and thus  $\min\{q_1, q_2\} \geq \frac{N}{\Lambda Q}$ . We also have

$$|n| r_1 r_2 \delta \leq r_1 r_2 \delta \cdot \frac{q_1 q_2 \delta \Lambda}{N} \leq \frac{\Lambda Q^2}{N} \ll \Lambda.$$

To estimate

$$S_{r_1, r_2, \delta, n}(Q) := \sum_{\substack{\min\{q_1, q_2\} > \frac{N}{\Lambda Q} \\ q_1 \in [1, \frac{Q}{r_1 \delta}], q_2 \in [1, \frac{Q}{r_2 \delta}] \\ (q_1, q_2) = 1}} \psi\left(\frac{nN}{q_1 q_2 \delta}\right),$$

we take  $f(x, y) := \psi(\frac{nN}{\delta xy})$ ,  $\Omega := \{(x, y) : x \leq \frac{Q}{r_1 \delta}, y \leq \frac{Q}{r_2 \delta}, \min\{x, y\} \geq \frac{N}{\Lambda Q}\}$ , and apply the following:

**Lemma 3** (Lemma 2 and Corollary 1 in [1]). *Suppose that  $\Omega \subseteq [1, R]^2$  is a region with rectifiable boundary and  $f \in C^1(\Omega)$  with  $Df = \left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$  and  $\|\cdot\|_\infty$  denoting the sup norm on  $\Omega$ . Then*

$$\sum_{\substack{(m, n) \in \Omega \\ (m, n) = 1}} f(m, n) = \frac{6}{\pi^2} \iint_{\Omega} f(x, y) dx dy + \mathcal{E}_{f, \Omega, R},$$

with

$$\mathcal{E}_{f, \Omega, R} \ll \|Df\|_\infty \text{Area}(\Omega) \log R + \|f\|_\infty (R + \text{length}(\partial\Omega) \log R).$$

Furthermore, if  $\Omega$  is also convex, then

$$\mathcal{E}_{f, \Omega, R} \ll \|Df\|_\infty \text{Area}(\Omega) \log R + \|f\|_\infty R \log R.$$

It is plain that  $|\psi(x)| \leq \frac{1}{\pi^2 x^2}$  and  $|\psi'(x)| \leq \frac{4}{\pi x^2}$ , thus on  $\Omega$  we have  $|f(x, y)| \leq \frac{\delta^2 x^2 y^2}{n^2 N^2} \leq \frac{Q^4}{N^2} \cdot \frac{1}{r_1^2 r_2^2 \delta^2 n^2}$  and  $|(Df)(x, y)| \leq \frac{2\delta^2 x^2 y^2}{n^2 N^2} \cdot \frac{|n|N}{\delta} \left(\frac{1}{x^2 y} + \frac{1}{xy^2}\right) \ll \frac{\delta(x+y)}{|n|N} \ll \frac{1}{|n|Q}.$

This yields

$$S_{r_1, r_2, \delta, n}(Q) = \frac{6}{\pi^2} \iint_{\substack{x \leq \frac{Q}{r_1 \delta}, y \leq \frac{Q}{r_2 \delta} \\ \min\{x, y\} \geq \frac{N}{\Lambda Q}}} \psi\left(\frac{nN}{\delta xy}\right) dx dy + E_{r_1, r_2, \delta, n}(Q), \quad (2.9)$$

with error terms

$$E_{r_1, r_2, \delta, n}(Q) \ll \frac{1}{|n|Q} \cdot \frac{Q^2}{r_1^2 r_2^2 \delta^2} \log Q + \frac{Q \log Q}{r_1^2 r_2^2 \delta^2 n^2},$$

summing up in  $\mathfrak{M}_Q^I(2)$  to

$$\mathcal{E}(Q) = \frac{1}{N} \sum_{\substack{|n|, r_1, r_2, \delta \geq 1 \\ |n| r_1 r_2 \delta \leq \Lambda}} \delta E_{r_1, r_2, \delta, n}(Q) \ll \frac{\log Q}{Q} \sum_{\substack{n, r_1, r_2, \delta \geq 1 \\ n r_1 r_2 \delta \leq \Lambda}} \frac{1}{r_1^2 r_2^2 \delta n} \ll \frac{(\log Q)^3}{Q}. \quad (2.10)$$

With the change of variables  $(x, y) = (Qu, Qv)$  the main term in (2.9) becomes

$$\frac{6Q^2}{\pi^2} \iint_{[\frac{N}{\Lambda Q^2}, \frac{1}{r_1 \delta}] \times [\frac{N}{\Lambda Q^2}, \frac{1}{r_2 \delta}]} \psi\left(\frac{nN}{Q^2 \delta uv}\right) du dv. \quad (2.11)$$

When  $0 < \min\{u, v\} < \frac{N}{\Lambda Q^2}$  and  $\max\{\delta u, \delta v\} \leq 1$  we have  $\frac{|n|N}{Q^2 uv \delta} \geq \frac{N}{Q^2} \cdot \frac{1}{uv \delta} \geq \frac{N}{Q^2} \cdot \frac{1}{\min\{u, v\}} > \Lambda$ , so  $\psi_\Lambda\left(\frac{nN}{Q^2 \delta uv}\right) = 0$ . Thus the expression in (2.11) amounts to

$$\frac{6Q^2}{\pi^2} \iint_{[0, \frac{1}{r_1 \delta}] \times [0, \frac{1}{r_2 \delta}]} \psi\left(\frac{nN}{Q^2 \delta uv}\right) du dv + O\left(Q^2 \cdot \frac{1}{\Lambda^2} \cdot \frac{1}{r_1 r_2 \delta^2}\right), \quad (2.12)$$

with the total contribution of the error term to  $\mathfrak{M}_Q^I(2)$  being

$$\ll \frac{1}{N} \sum_{\substack{n, r_1, r_2, \delta \geq 1 \\ n r_1 r_2 \delta \leq \Lambda}} \frac{1}{r_1 r_2 \delta^2} \ll \frac{Q \log^2 Q}{N} \ll \mathcal{E}(Q).$$

Using (2.11), (2.12),  $\psi(x) = \psi(-x)$  and the change of variable  $(u, v) \mapsto (u, \lambda)$  with  $\lambda = \frac{N}{Q^2} \cdot \frac{|n|}{uv \delta}$ , the main term in (2.9) becomes, up to an additive error of order  $O(\frac{1}{r_1 r_2 \delta^2})$ ,

$$\frac{6N}{\pi^2} \cdot \frac{|n|}{\delta} \int_0^{\frac{1}{r_1 \delta}} du \int_{\frac{N}{Q^2} \cdot \frac{r_2 |n|}{u}}^{\Lambda} \psi(\lambda) \frac{d\lambda}{u \lambda^2} = \frac{6N|n|}{\pi^2 \delta} \cdot \int_{\frac{N}{Q^2} \cdot |n| r_1 r_2 \delta}^{\Lambda} \int_{\frac{N}{Q^2} \cdot \frac{r_2 |n|}{\lambda}}^{\frac{1}{r_1 \delta}} \frac{\psi(\lambda)}{\lambda^2 u} du d\lambda.$$

We infer that

$$\mathfrak{M}_Q^{II}(2) = \frac{12}{\pi^2} \sum_{\substack{n, r_1, r_2, \delta \geq 1 \\ nr_1 r_2 \delta \leq \frac{\Lambda Q^2}{N}}} \mu(r_1) \mu(r_2) n \int_{\frac{N}{Q^2} \cdot nr_1 r_2 \delta}^{\Lambda} \frac{\psi(\lambda)}{\lambda^2} \log \left( \frac{Q^2}{N} \cdot \frac{\lambda}{nr_1 r_2 \delta} \right) d\lambda + \mathcal{E}(Q).$$

Taking  $K = nr_1 r_2 \delta \in [1, \frac{\Lambda Q^2}{N}]$  and using

$$\sum_{\substack{n, r_1, r_2 \geq 1 \\ nr_1 r_2 | K}} \mu(r_1) \mu(r_2) n = \varphi(K)$$

and  $|\psi(x)| \leq \frac{1}{x^2}$ , we infer

$$\begin{aligned} \mathfrak{M}_Q^{II}(2) &= \frac{12}{\pi^2} \sum_{K \in [1, \frac{\Lambda Q^2}{N}]} \varphi(K) \int_{\frac{N}{Q^2} K}^{\Lambda} \frac{\psi(\lambda)}{\lambda^2} \log \left( \frac{Q^2}{N} \cdot \frac{\lambda}{K} \right) d\lambda + \mathcal{E}(Q) \\ &= \frac{12}{\pi^2} \int_0^{\Lambda} \frac{\psi(\lambda)}{\lambda^2} \sum_{K \in [1, \frac{\lambda Q^2}{N}]} \varphi(K) \max \left\{ 0, \log \left( \frac{Q^2}{N} \cdot \frac{\lambda}{K} \right) \right\} d\lambda + \mathcal{E}(Q) \\ &= \frac{18}{\pi^4} \cdot \frac{Q^4}{N^2} \int_0^{\Lambda} \psi(\lambda) g_2 \left( \frac{3}{\pi^2} \cdot \frac{Q^2}{N} \lambda \right) d\lambda + \mathcal{E}(Q) \\ &= \frac{18}{\pi^4} \cdot \frac{Q^4}{N^2} \int_0^{\infty} \psi(\lambda) g_2 \left( \frac{3}{\pi^2} \cdot \frac{Q^2}{N} \lambda \right) d\lambda + O(\Lambda^{-1}) + \mathcal{E}(Q) \\ &= \frac{18}{\pi^4} \cdot \frac{Q^2}{N} \int_0^{\infty} \psi \left( \frac{N}{Q^2} x \right) g_2 \left( \frac{3}{\pi^2} x \right) dx + O \left( \frac{(\log Q)^3}{Q} \right), \end{aligned} \tag{2.13}$$

with the function  $g_2$  defined as in [2] by

$$g_2 \left( \frac{3}{\pi^2} u \right) = \frac{2\pi^2}{3u^2} \sum_{K \in [1, u]} \varphi(K) \log \left( \frac{u}{K} \right), \tag{2.14}$$

being continuous, supported on  $[\frac{3}{\pi^2}, \infty)$ , with  $\|g_2'\|_{\infty} < \infty$  and  $g_2(x) = 1 + O(\frac{1}{x})$  as  $x \rightarrow \infty$ .

Using the dominated convergence theorem we conclude that,

**Proposition 4.** *If  $N \sim \alpha Q^2$  for some  $\alpha > 0$  as  $Q \rightarrow \infty$ , then*

$$\lim_Q \mathfrak{M}_Q(2) = M_2(\alpha) := \frac{3}{\pi^2 \alpha} + \left( \frac{3}{\pi^2} \right)^2 \cdot \frac{2}{\alpha} \int_0^{\infty} \text{sinc}^2(\pi \alpha u) g_2 \left( \frac{3}{\pi^2} u \right) du.$$

**Remark 5.** Since  $|\psi(x)| \leq \frac{1}{x^2}$ ,  $\psi \in C^1([0, \infty))$  and  $\|g_2\|_{\infty} < 1$ , it is easily seen, by truncating the integral in Proposition 4 at  $Q^{\beta/2}$ , that if

$$N = \alpha Q^2 (1 + O(Q^{-\beta}))$$

for some  $\beta > 0$ , then

$$\mathfrak{M}_Q(2) = M_2(\alpha) + O(Q^{-\beta/2}).$$

Using a different description of  $g_2(x)$ , due to R. R. Hall and presented in [2], we also see that this main term matches the expression given in Theorem 1 (we however do not need this, since we reprove Proposition 4 in Section 5, when dealing with the general case of all  $\ell \geq 2$ ).

**2.2. Comparison with Ramaré's main term.** Ramaré's estimate of  $\mathfrak{M}_Q(2)$  produced the following main term (see the formula between (46) and (47) and formula (47) in [14]):

$$\mathfrak{M}_Q(2) \sim \frac{|\mathcal{F}_Q|}{N} + \frac{Q^4}{N^2} \cdot \mathfrak{h}\left(\frac{N}{Q^2}\right), \quad (2.15)$$

where

$$\mathfrak{h}(x) = \frac{6}{\pi^3 i} \int_{-\frac{1}{8}-i\infty}^{-\frac{1}{8}+i\infty} \frac{x^s}{s(s+1)(2-s)^2} \cdot \frac{\zeta(s)}{\zeta(2-s)} ds. \quad (2.16)$$

Employing formula (4.15) in [2] we can write

$$\begin{aligned} g_2\left(\frac{3}{\pi^2}u\right) &= \frac{2\pi^2}{3u^2} \cdot \frac{1}{2\pi i} \int_{\frac{17}{8}-i\infty}^{\frac{17}{8}+i\infty} \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{u^s}{s^2} ds \\ &= \frac{2\pi^2}{3u^2} \cdot \frac{1}{2\pi i} \int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(s)}{\zeta(1+s)} \cdot \frac{u^{1+s}}{(1+s)^2} ds. \end{aligned} \quad (2.17)$$

Employing also Fubini we infer

$$\begin{aligned} I_\alpha &:= \frac{3}{\pi^2} \int_0^\infty \text{sinc}^2(\alpha u) g_2\left(\frac{3}{\pi^2}u\right) du \\ &= \frac{1}{\pi i} \int_0^\infty \text{sinc}^2(\alpha u) \int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{1}{u^2} \cdot \frac{u^{1+s}}{(1+s)^2} \cdot \frac{\zeta(s)}{\zeta(1+s)} ds du \\ &= \frac{1}{\pi i \alpha^2} \int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(s)}{(1+s)^2 \zeta(1+s)} \int_0^\infty \frac{\sin^2(\alpha u)}{u^{3-s}} du ds. \end{aligned} \quad (2.18)$$

Employing the identity (cf. formula 3.823 page 454 in [10])

$$\int_0^\infty \frac{\sin^2 x}{x^\nu} dx = -2^{\nu-2} \Gamma(1-\nu) \cos\left(\frac{(1-\nu)\pi}{2}\right) \quad \text{if } 1 < \text{Re } \nu < 3, \quad (2.19)$$

we find

$$\int_0^\infty \frac{\sin^2(\alpha u)}{u^{3-s}} du = \alpha^{2-s} 2^{1-s} \Gamma(s-2) \cos\left(\frac{\pi s}{2}\right) = \frac{2\alpha^2}{(2\alpha)^s} \cdot \frac{\Gamma(s) \cos\left(\frac{\pi s}{2}\right)}{(s-2)(s-1)},$$

which we insert into (2.18) to derive

$$I_\alpha = \frac{2}{\pi i} \int_{9/8-i\infty}^{9/8+i\infty} \frac{(2\alpha)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right)}{(2-s)(1-s)(1+s)^2} \cdot \frac{\zeta(s)}{\zeta(1+s)} ds.$$

The functional equation

$$\zeta(s) = \frac{\pi \zeta(1-s)}{(2\pi)^{1-s} \sin\left(\frac{\pi(1-s)}{2}\right) \Gamma(s)}$$

and the change of variables  $s \mapsto 1-s$  provide

$$\begin{aligned} I_\alpha &= \frac{1}{\pi i} \int_{9/8-i\infty}^{9/8+i\infty} \left(\frac{\pi}{\alpha}\right)^s \frac{1}{(2-s)(1-s)(1+s)^2} \cdot \frac{\zeta(1-s)}{\zeta(1+s)} ds \\ &= \frac{1}{\pi i} \int_{-1/8-i\infty}^{-1/8+i\infty} \left(\frac{\pi}{\alpha}\right)^{1-s} \frac{1}{s(1+s)(2-s)^2} \cdot \frac{\zeta(s)}{\zeta(2-s)} ds. \end{aligned}$$

Finally, inserting this into (2.13) we infer

$$\begin{aligned} \mathfrak{M}_Q^{II}(2) &\sim \frac{Q^2}{N} \cdot \frac{6}{\pi^2} I_{\pi N/Q^2} \\ &= \frac{Q^2}{N} \cdot \frac{6}{\pi^2} \cdot \frac{1}{\pi i} \int_{-1/8-i\infty}^{-1/8+i\infty} \frac{(Q^2/N)^{1-s}}{s(s+1)(2-s)^2} \cdot \frac{\zeta(s)}{\zeta(2-s)} ds \\ &= \frac{Q^4}{N^2} \cdot \frac{6}{\pi^3 i} \int_{-1/8-i\infty}^{-1/8+i\infty} \frac{(N/Q^2)^s}{s(s+1)(2-s)^2} \cdot \frac{\zeta(s)}{\zeta(2-s)} ds \\ &= \frac{Q^4}{N^2} \cdot \mathfrak{h}\left(\frac{N}{Q^2}\right). \end{aligned} \tag{2.20}$$

From (2.7) and (2.20) we notice that our main term  $M_2(\alpha)$  in the asymptotic formula for  $\mathfrak{M}_Q(2)$  given in Proposition 4 coincides with the one in [14, Theorem 1.1].

### 3. SMOOTHING OF $\mathfrak{M}_Q(\ell)$

As seen in the previous section, when dealing with  $\mathfrak{M}_Q(2)$  it is possible to proceed directly without smoothing the characteristic function  $H_N$ . However, smoothing becomes necessary for  $\ell \geq 3$ , due to the accumulations of terms. In this section we show that this can be efficiently achieved employing the large sieve inequality.

Let  $\delta \in (0, 1)$ . We pick a function  $f_\delta \in C_c^\infty(\mathbb{R})$  such that  $0 \leq f_\delta \leq 1$ ,  $f_\delta \equiv 1$  on the interval  $[\delta, 1]$ , and  $\text{supp } f_\delta = [0, 1 + \delta]$ . Consider  $\Theta = (\theta_1, \dots, \theta_\ell) \in \mathcal{F}_Q^\ell$ , the function

$$\begin{aligned} h_\Theta(x_1, \dots, x_\ell) &:= e(x_1(\theta_1 - \theta_\ell) + x_2(\theta_2 - \theta_1) + \dots + x_\ell(\theta_\ell - \theta_{\ell-1})) \\ &= e((x_1 - x_2)\theta_1 + (x_2 - x_3)\theta_2 + \dots + (x_\ell - x_1)\theta_\ell), \end{aligned} \tag{3.1}$$

and its smoothed form

$$h_{\delta;\Theta}(x_1, \dots, x_\ell) := h_\Theta(x_1, \dots, x_\ell) f_\delta\left(\frac{x_1}{N}\right) \cdots f_\delta\left(\frac{x_\ell}{N}\right). \quad (3.2)$$

In this section we will show that the large sieve inequality allows us to replace  $\mathfrak{M}_Q(\ell)$  by its smoothed version

$$\begin{aligned} \mathfrak{M}_{Q;\delta}(\ell) &:= \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ n_1, \dots, n_\ell \in \mathbb{Z}}} h_{\delta;\Theta}(n_1, \dots, n_\ell) \\ &= \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ 0 < n_1, \dots, n_\ell < (1+\delta)N}} h_{\delta;\Theta}(n_1, \dots, n_\ell). \end{aligned} \quad (3.3)$$

On the other hand we have

$$\mathfrak{M}_Q(\ell) = \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ 0 < n_1, \dots, n_\ell < (1+\delta)N}} h_\Theta(n_1, \dots, n_\ell) \mathbf{1}_{(0,1]}\left(\frac{n_1}{N}\right) \cdots \mathbf{1}_{(0,1]}\left(\frac{n_\ell}{N}\right),$$

where  $\mathbf{1}_{\mathcal{S}}$  denotes the characteristic function of a set  $\mathcal{S}$ .

For disjoint subsets  $\mathcal{M}, \mathcal{A}, \mathcal{B}$  of  $[1, \ell]$  consider

$$\begin{aligned} \mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)} &:= \frac{1}{N^{\ell+1}} \sum_{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q} \sum_{\substack{n_1, \dots, n_\ell \\ \{j: 0 < n_j < \delta N\} = \mathcal{A} \\ \{k: N < n_k < (1+\delta)N\} = \mathcal{B} \\ \{i: \delta N \leq n_i \leq N\} = \mathcal{M}}} h_{\delta;\Theta}(n_1, \dots, n_\ell), \\ \mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(2)} &:= \frac{1}{N^{\ell+1}} \sum_{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q} \sum_{\substack{n_1, \dots, n_\ell \\ \{j: 0 < n_j < \delta N\} = \mathcal{A} \\ \{k: N < n_k < (1+\delta)N\} = \mathcal{B} \\ \{i: \delta N \leq n_i \leq N\} = \mathcal{M}}} h_\Theta(n_1, \dots, n_\ell) \mathbf{1}_{(0,1]}\left(\frac{n_1}{N}\right) \cdots \mathbf{1}_{(0,1]}\left(\frac{n_\ell}{N}\right). \end{aligned}$$

If  $\mathcal{B} \neq \emptyset$ , then  $\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(2)} = 0$ . We have

$$\mathfrak{M}_Q(\ell) = \sum_{\mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{M} = [1, \ell]} \mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(2)}, \quad \mathfrak{M}_{Q;\delta}(\ell) = \sum_{\mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{M} = [1, \ell]} \mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)}.$$

Employing  $\mathcal{I}_{[1, \ell], \emptyset, \emptyset}^{(1)} = \mathcal{I}_{[1, \ell], \emptyset, \emptyset}^{(2)}$  we can write

$$\mathfrak{M}_Q(\ell) - \mathfrak{M}_{Q;\delta}(\ell) = \sum_{\substack{\mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{M} = [1, \ell] \\ \mathcal{M} \neq [1, \ell]}} (\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(2)} - \mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)}). \quad (3.4)$$

We will now bound the contribution of each  $\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)}$  with  $\mathcal{M} \neq [1, \ell]$  using the large sieve inequality. The contribution of  $\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(2)}$  will be dealt with in identical manner

by taking  $f_\delta(\frac{n}{N}) = \mathbf{1}_{(0,1]}(\frac{n}{N})$  in the argument that is about to follow. For this reason we will only write down the argument for  $\mathcal{I}_{\mathcal{M},\mathcal{A},\mathcal{B}}^{(1)}$ . Consider

$$x_{n,\theta} := e(n\theta) \sqrt{f_\delta\left(\frac{n}{N}\right)}, \quad 0 < n < (1+\delta)N, \theta \in \mathcal{F}_Q,$$

and the rectangular matrices  $A, B \in M_{[\delta N], |\mathcal{F}_Q|}(\mathbb{C})$ ,  $M \in M_{N-[\delta N], |\mathcal{F}_Q|}(\mathbb{C})$  with entries  $x_{n,\theta}$  where  $\theta \in \mathcal{F}_Q$  and  $0 < n < \delta N$  for  $A$ ,  $N < n < (1+\delta)N$  for  $B$ , and  $\delta N \leq n \leq N$  for  $M$ , respectively. Clearly  $A^*A, B^*B, M^*M$  are  $|\mathcal{F}_Q| \times |\mathcal{F}_Q|$  matrices with

$$\begin{aligned} (A^*A)_{\theta',\theta''} &= \sum_{0 < n < \delta N} e(n(\theta'' - \theta')) f_\delta\left(\frac{n}{N}\right), \\ (B^*B)_{\theta',\theta''} &= \sum_{N < n < (1+\delta)N} e(n(\theta'' - \theta')) f_\delta\left(\frac{n}{N}\right), \\ (M^*M)_{\theta',\theta''} &= \sum_{\delta N \leq n \leq N} e(n(\theta'' - \theta')) f_\delta\left(\frac{n}{N}\right). \end{aligned} \tag{3.5}$$

Writing

$$\begin{aligned} M &= \text{diag} \left( \sqrt{f_\delta\left(\frac{n}{N}\right)} \right)_{\delta N \leq n \leq N} \cdot \left( e(n\theta) \right)_{\substack{\delta N \leq n \leq N \\ \theta \in \mathcal{F}_Q}}, \\ A &= \text{diag} \left( \sqrt{f_\delta\left(\frac{n}{N}\right)} \right)_{0 < n < \delta N} \cdot \left( e(n\theta) \right)_{\substack{0 < n < \delta N \\ \theta \in \mathcal{F}_Q}}, \\ B &= \text{diag} \left( \sqrt{f_\delta\left(\frac{n}{N}\right)} \right)_{N < n < (1+\delta)N} \cdot \left( e(n\theta) \right)_{\substack{N < n < (1+\delta)N \\ \theta \in \mathcal{F}_Q}} \end{aligned}$$

and employing  $0 \leq f_\delta \leq 1$ , the large sieve inequality provides

$$\|M^*M\| \leq N + Q^2 \quad \text{and} \quad \max \{ \|A^*A\|, \|B^*B\| \} \leq \delta N + Q^2. \tag{3.6}$$

Since  $\max\{\text{rank}(A), \text{rank}(B)\} \leq \delta N$ , we have  $\max\{\text{rank}(A^*A), \text{rank}(B^*B)\} \leq \delta N$ . Since  $\text{rank}(X_1 \cdots X_\ell) \leq \min\{\text{rank}(X_1), \dots, \text{rank}(X_\ell)\}$ , we infer

$$\text{rank} \left( \prod_{r=1}^{\ell} (M^*M)^{\alpha_r} (A^*A)^{\beta_r} (B^*B)^{\gamma_r} \right) \leq \delta N \tag{3.7}$$

whenever  $\alpha_r, \beta_r, \gamma_r \in \{0, 1\}$  and there exists  $r_0 \in [1, \ell]$  such that  $\beta_{r_0} > 0$  or  $\gamma_{r_0} > 0$ .

On the other hand, setting  $S(r) := \mathcal{A}, \mathcal{B}$  or  $\mathcal{M}$  according to whether  $r \in \mathcal{A}$ ,  $r \in \mathcal{B}$  or  $r \in \mathcal{M}$  and using (3.5), we see that the  $(\theta', \theta'')$ -entry of the product  $\prod_{r=1}^{\ell} (M^*M)^{\mathbf{1}_{\mathcal{M}(r)}} (A^*A)^{\mathbf{1}_{\mathcal{A}(r)}} (B^*B)^{\mathbf{1}_{\mathcal{B}(r)}}$  of  $\ell$  matrices of the form  $M^*M$ ,  $A^*A$  or  $B^*B$



is given by

$$\sum_{\substack{\theta_1, \dots, \theta_{\ell-1} \in \mathcal{F}_Q \\ 0 < n_1, \dots, n_\ell < (1+\delta)N \\ n_r \in S(r), \forall r \in [1, \ell]}} e(n_1(\theta_1 - \theta')) f_\delta\left(\frac{n_1}{N}\right) e(n_2(\theta_2 - \theta_1)) f_\delta\left(\frac{n_2}{N}\right) \\ \cdots e(n_{\ell-1}(\theta_{\ell-1} - \theta_{\ell-2})) f_\delta\left(\frac{n_{\ell-1}}{N}\right) e(n_\ell(\theta'' - \theta_{\ell-1})) f_\delta\left(\frac{n_\ell}{N}\right).$$

In conjunction with the definition of  $\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)}$ , (3.1) and (3.2) and setting  $\theta_0 = \theta_\ell = \theta' = \theta''$ , this further leads to

$$\begin{aligned} \mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)} &= \frac{1}{N^{\ell+1}} \sum_{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q} \sum_{\substack{\{j: 0 < n_j < \delta N\} = \mathcal{A} \\ \{k: N < n_k < (1+\delta)N\} = \mathcal{B} \\ \{i: \delta N \leq n_i \leq N\} = \mathcal{M}}} \prod_{r=1}^{\ell} e(n_r(\theta_r - \theta_{r-1})) f_\delta\left(\frac{n_r}{N}\right) \\ &= \frac{1}{N^{\ell+1}} \text{Tr} \left( \prod_{r=1}^{\ell} (M^* M)^{\mathbf{1}_{\mathcal{M}(r)}} (A^* A)^{\mathbf{1}_{\mathcal{A}(r)}} (B^* B)^{\mathbf{1}_{\mathcal{B}(r)}} \right). \end{aligned} \quad (3.8)$$

Employing (3.8), the inequality  $\text{Tr}(X) \leq \text{rank}(X) \|X\|$  for any square matrix  $X$ , and inequalities (3.6) and (3.7), we infer

$$|\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(1)}| \leq \frac{1}{N^{\ell+1}} \cdot \delta N (N + Q^2)^\ell \ll_\ell \delta \quad \text{whenever } \mathcal{M} \neq [1, \ell]. \quad (3.9)$$

A similar bound holds on  $\mathcal{I}_{\mathcal{M}, \mathcal{A}, \mathcal{B}}^{(2)}$ , with  $f_\delta(\frac{n}{N}) = \mathbf{1}_{(0,1]}(\frac{n}{N})$  above, hence (3.4) and (3.9) yield

$$\mathfrak{M}_Q(\ell) = \mathfrak{M}_{Q, \delta}(\ell) + O_\ell(\delta). \quad (3.10)$$

#### 4. ANALYSIS OF THE MAIN TERM $M_\ell(\alpha)$

Fix  $k = \ell - 1 \geq 1$  and a constant  $\alpha > 0$ . For every  $A, B \in \mathbb{Z}$ ,  $A^2 + B^2 \neq 0$ , consider the function  $\beta_{A, B, \alpha}$  defined by

$$\beta_{A, B, \alpha}(x, y) := \frac{\alpha B}{y(Ay - Bx)}. \quad (4.1)$$

Let  $\mathfrak{F}$  denote the set of functions  $F : \mathbb{R} \rightarrow \mathbb{C}$  that satisfy

$$F(0) = 1, \quad F(-u) = F(u), \quad |F(u)| \leq \min \left\{ 1, \frac{1}{|u|} \right\}, \quad \forall u \in \mathbb{R}.$$

Denote

$$\psi_F(x_1, \dots, x_k) := \begin{cases} F(-x_1) \prod_{i=1}^{k-1} F(x_i - x_{i+1}) F(x_k) & \text{if } k \geq 2, \\ F(-x_1) F(x_1) & \text{if } k = 1. \end{cases} \quad (4.2)$$

For every  $\mathbf{A} = (A_1, \dots, A_k), \mathbf{B} = (B_1, \dots, B_k) \in \mathbb{Z}^k$ , consider the function in two variables

$$\Psi_{F;\mathbf{A},\mathbf{B},\alpha}(x, y) := \Psi_F(\beta_{A_1, B_1, \alpha}(x, y), \dots, \beta_{A_k, B_k, \alpha}(x, y)), \quad (4.3)$$

and the set  $\mathcal{D}_{\mathbf{A}, \mathbf{B}}$  defined in (1.3). Consider also

$$I_{k, \delta, \alpha}(F) := \sum_{\substack{\mathbf{A}, \mathbf{B} \in \mathbb{Z}^k \\ (A_i, B_i) = 1, \forall i \\ A_i^2 + B_i^2 \neq 0, \forall i}} \max_{i \in [1, k]} \{|A_i|^\delta, |B_i|^\delta\} \iint_{\mathcal{D}_{\mathbf{A}, \mathbf{B}}} |\Psi_{F;\mathbf{A}, \mathbf{B}, \alpha}(x, y)| dx dy \in [0, \infty].$$

Recall that we take  $(A, 0) = |A|$ , so if  $B_i = 0$  for some  $i$  in some non-zero term of  $I_{k, \delta, \alpha}(F)$ , then  $|A_i| \geq 1$ .

The aim of this section is to prove the following:

**Proposition 6.** *There exists  $\delta = \delta_\ell > 0$  such that for every  $\alpha > 0$  we have*

$$\sup_{F \in \mathfrak{F}} I_{k, \delta, \alpha}(F) \ll_k \alpha^{-k-1} + 1 < \infty.$$

In particular this establishes part (ii) in Theorem 1. Before starting the proof of Proposition 6, we note its subsequent important consequence, which gives part (iii) in Theorem 1.

**Corollary 7.** *Let  $0 < \gamma_1 < \gamma_2$  be given. With  $M_\ell(\alpha)$  as in (1.4), and uniformly in  $\alpha, \beta \in [\gamma_1, \gamma_2]$ , we have*

$$|M_\ell(\alpha) - M_\ell(\beta)| \ll_{\gamma_1, \gamma_2} |\alpha - \beta|^{\kappa_\ell}$$

for some exponent  $\kappa_\ell \in (0, 1)$ .

*Proof.* Consider

$$G_F(\alpha, \ell) = \sum_{\substack{\mathbf{A}, \mathbf{B} \in \mathbb{Z}^k \\ (A_i, B_i) = 1, \forall i}} \iint_{\mathcal{D}_{\mathbf{A}, \mathbf{B}}} \Psi_{F;\mathbf{A}, \mathbf{B}, \alpha}(x, y) dx dy.$$

If  $F$  is continuous, then Proposition 6 and the dominated convergence theorem show that  $G_F(\alpha, \ell)$  is continuous in  $\alpha$ . Note that  $M_\ell(\alpha) = \frac{6}{\pi^2 \alpha} G_{\widehat{\chi}_{[0,1]}}(\alpha, \ell)$ . Since  $\widehat{\chi}_{[0,1]}$  is continuous it follows that  $M_\ell(\alpha)$  is also continuous. In order to establish the stronger bound note that for arbitrary functions  $f$  and  $g$  we have,

$$|(f \cdot g)(\alpha) - (f \cdot g)(\beta)| \leq |f(\alpha) - f(\beta)| \cdot \|g\|_\infty + |g(\alpha) - g(\beta)| \cdot \|f\|_\infty. \quad (4.4)$$

Since  $\alpha \mapsto \frac{1}{\alpha}$  is Lipschitz continuous on the interval  $[\gamma_1, \gamma_2]$ , it is therefore enough to show that  $G_{\widehat{\chi}_{[0,1]}}(\alpha, \ell)$  satisfies the bound

$$|G_{\widehat{\chi}_{[0,1]}}(\alpha, \ell) - G_{\widehat{\chi}_{[0,1]}}(\beta, \ell)| \ll_{\gamma_1, \gamma_2} |\alpha - \beta|^{\kappa_\ell}$$

for some exponent  $\kappa_\ell > 0$ .

Using Proposition 6 we can truncate the expressions defining  $G_{\widehat{\chi}_{[0,1]}}(\alpha, \ell)$  and  $G_{\widehat{\chi}_{[0,1]}}(\beta, \ell)$  at  $\max_i\{|A_i|, |B_i|\} \leq |\alpha - \beta|^{-\eta}$  at the price of an error term  $\ll |\alpha - \beta|^{\eta\delta_\ell}$ . That is,  $G_{\widehat{\chi}_{[0,1]}}(\alpha, \ell) - G_{\widehat{\chi}_{[0,1]}}(\beta, \ell)$  is equal to

$$\sum_{\substack{|A_i| \leq |\alpha - \beta|^{-\eta} \\ 0 < |B_i| \leq |\alpha - \beta|^{-\eta} \\ (A_i, B_i) = 1, \forall i}} \iint_{\mathcal{D}_{\mathbf{A}, \mathbf{B}}} (\Psi_{\widehat{\chi}_{[0,1]}, \mathbf{A}, \mathbf{B}, \alpha}(x, y) - \Psi_{\widehat{\chi}_{[0,1]}, \mathbf{A}, \mathbf{B}, \beta}(x, y)) dx dy + O(|\alpha - \beta|^{\eta\delta_\ell}),$$

where

$$\Psi_{\widehat{\chi}_{[0,1]}, \mathbf{A}, \mathbf{B}, \alpha}(x, y) := \Psi_{\widehat{\chi}_{[0,1]}}(\beta_{A_1, B_1, \alpha}(x, y), \dots, \beta_{A_{\ell-1}, B_{\ell-1}, \alpha}(x, y)).$$

The product of bounded Lipschitz continuous functions is Lipschitz continuous by (4.4) and therefore,

$$|\Psi_{\widehat{\chi}_{[0,1]}, \mathbf{A}, \mathbf{B}, \alpha}(x, y) - \Psi_{\widehat{\chi}_{[0,1]}, \mathbf{A}, \mathbf{B}, \beta}(x, y)| \ll_{\gamma_1, \gamma_2} |\alpha - \beta|.$$

Combining the previous three equations we conclude that,

$$|G_{\widehat{\chi}_{[0,1]}}(\alpha, \ell) - G_{\widehat{\chi}_{[0,1]}}(\beta, \ell)| \ll_{\gamma_1, \gamma_2} |\alpha - \beta|^{\kappa_\ell},$$

where  $\kappa_\ell := \min(1 - 2(2\ell - 2)\eta, \eta\delta_\ell) \in (0, 1)$ . Taking  $\eta > 0$  sufficiently small shows that  $\kappa_\ell \in (0, 1)$ .  $\square$

We will require two lemmas for the proof of Proposition 6. First we record a simple bound for  $\Psi_{F, \mathbf{A}, \mathbf{B}}$ :

**Lemma 8.** *Let  $I := \{i \in [1, k-1] : A_i B_{i+1} - A_{i+1} B_i \neq 0\}$ . Suppose that  $B_1 \neq 0$  and  $B_k \neq 0$ . Then, for every  $\alpha > 0$  and  $(x, y) \in \mathcal{D}_{\mathbf{A}, \mathbf{B}}$  we have*

$$\sup_{F \in \mathfrak{F}} |\Psi_{F, \mathbf{A}, \mathbf{B}}(x, y)| \leq \frac{y^2}{\alpha^{|I|+2} |B_1 B_k|} \prod_{i \in I} \frac{|A_i y - B_i x|}{|A_i B_{i+1} - A_{i+1} B_i|}.$$

*Proof.* The first inequality follows from the bounds

$$|\psi_F(x_1, \dots, x_k)| \leq \frac{1}{|x_1 x_k|} \prod_{i \in I} \frac{1}{|x_i - x_{i+1}|}, \quad |A_i y - B_i x| \leq 1,$$

and from equality

$$\beta_{A_i, B_i, \alpha}(x, y) - \beta_{A_{i+1}, B_{i+1}, \alpha}(x, y) = \frac{\alpha(A_{i+1} B_i - A_i B_{i+1})}{(A_i y - B_i x)(A_{i+1} y - B_{i+1} x)}. \quad (4.5)$$

$\square$

Our argument will rely crucially on the (non-disjoint) dy-adic set equality  $\mathbb{N} = \bigcup_{a \in \mathbb{N}_0} [2^a - 1, 2^{a+1}]$  and on the following “counting lemma”:

**Lemma 9.** *Given  $\mathbf{D} = (D_1, \dots, D_{k-1}) \in \mathbb{Z}^{2k-2}$ ,  $\mathbf{a} = (a_1, \dots, a_k)$ ,  $\mathbf{b} = (b_1, \dots, b_k) \in \mathbb{N}_0^k$ , consider the set*

$$\mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}} := \left\{ (\mathbf{A}, \mathbf{B}) \in \mathbb{Z}^{2k} : \begin{array}{l} 2^{a_i} - 1 \leq |A_i| \leq 2^{a_i+1}, \quad 2^{b_i} - 1 \leq |B_i| \leq 2^{b_i+1}, \\ A_i B_{i+1} - A_{i+1} B_i = D_i, \quad (A_i, B_i) = 1, \forall i \end{array} \right\},$$

with  $\mathbf{A} = (A_1, \dots, A_k)$ ,  $\mathbf{B} = (B_1, \dots, B_k)$ .

For every  $x, y \in [0, 1]^2$  with  $x \leq y$  we have

$$\begin{aligned} \text{(i)} \quad & \sum_{(\mathbf{A}, \mathbf{B}) \in \mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} \prod_{i=1}^k \mathbf{1}_{0 < A_i y - B_i x \leq 1} \ll \frac{2^{b_1}}{y} \prod_{i=1}^{k-1} \min \{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \}. \\ \text{(ii)} \quad & \sum_{(\mathbf{A}, \mathbf{B}) \in \mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} 1 \ll 2^{a_1} 2^{b_1} \prod_{i=1}^{k-1} \min \{ 2^{b_{i+1}-b_i} + 1, 2^{a_{i+1}-a_i} + 1 \}. \end{aligned}$$

*Proof.* To prove (i), notice that given  $B_1$  and  $A_1$ , the condition  $A_1 B_2 - A_2 B_1 = D_1$  implies that

$$A_2 = x_1 + k_1 A_1 \quad \text{and} \quad B_2 = y_1 + k_1 B_1, \quad (4.6)$$

with  $(x_1, y_1)$  particular solution of  $A_1 y - B_1 x = D_1$  (so  $x_1 \equiv -\overline{B_1} D_1 \pmod{A_1}$  and  $y_1 \equiv \overline{A_1} D_1 \pmod{B_1}$ ) and  $k_1 \in \mathbb{Z}$ . In addition, since  $\frac{B_2 x}{y} \leq A_2 \leq \frac{B_2 x}{y} + \frac{1}{y}$  has to fit into an interval of length  $\leq \frac{1}{y}$  and  $|A_1| \geq 2^{a_1}$ , the number of choices of  $k_1$  for fixed  $(A_1, B_1)$  is

$$\ll (1/y) 2^{-a_1} + 1,$$

regardless of the choice of  $(x, y)$ . Repeating the same reasoning with  $B_2$  in place of  $A_2$  we see that  $B_2$  is also required to be contained in a short interval of length  $\leq \frac{1}{x}$ , since  $-\frac{A_2 y}{x} \leq -B_2 \leq -\frac{A_2 y}{x} + \frac{1}{x}$ . By the same argument it follows that the number of admissible choices for  $k_1$  is also

$$\ll (1/x) 2^{-b_1} + 1.$$

Therefore, regardless on  $(x, y)$ , the number of choices for  $k_1$  is

$$\ll \min \{ (1/y) 2^{-a_1} + 1, (1/x) 2^{-b_1} + 1 \}.$$

Continuing, we see that in general, given  $(A_i, B_i)$ , we have  $A_i B_{i+1} - A_{i+1} B_i = D_i$  and therefore  $(A_{i+1}, B_{i+1})$  is parameterized as

$$A_{i+1} = x_i + k_i A_i \quad \text{and} \quad B_{i+1} = y_i + k_i B_i,$$

with  $(x_i, y_i)$  fixed solution of  $A_i y - B_i x = D_i$  and  $k_i \in \mathbb{Z}$ . It follows that if we are given  $A_1$  and  $B_1$ , then the number of admissible choices for  $A_2, B_2, \dots, A_k, B_k$  is

$$\ll \prod_{i=1}^{k-1} \min \{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \}.$$

Finally, the number of choices for  $(A_1, B_1)$  is  $\leq 2^{b_1+1}(1 + \frac{1}{y}) \ll \frac{2^{b_1}}{y}$  regardless of  $(x, y)$  since  $|B_1| \leq 2^{b_1+1}$  and  $\frac{B_1 x}{y} \leq A_1 \leq \frac{B_1 x}{y} + \frac{1}{y}$ . This proves (i).

Part (ii) is proved in a similar way by first selecting  $(A_1, B_1)$  in at most  $2^{a_1+1}2^{b_1+1}$  ways, and then parameterizing as in (4.6). Since we require  $|B_2| \leq 2^{b_2+1}$  and  $|B_1| \geq 2^{b_1+1}$ , the number of choices for  $k_1$  is  $\ll 2^{b_2-b_1} + 1$ , and therefore the number of choices for  $(A_2, B_2)$  is  $\ll 2^{b_2-b_1} + 1$  as well. Now that we fixed  $(A_2, B_2)$ , it is seen in a similar way that the number of choices for  $(A_3, B_3)$  is  $\ll 2^{b_3-b_2} + 1$ , and so on, showing that the left hand side in (ii) is  $\ll 2^{a_1+b_1} \prod_{i=1}^{k-1} (2^{b_{i+1}-b_i} + 1)$ . Finally the roles of  $A_i$  and  $B_i$  can be interchanged to prove that the left hand side in (ii) is  $\ll 2^{a_1+b_1} \prod_{i=1}^{k-1} (2^{a_{i+1}-a_i} + 1)$ .  $\square$

With these two lemmas at hand, we are ready to start the proof of Proposition 6.

First we dispose of the easy case,  $k = 1$ . The constraint  $0 < A_1 y - B_1 x \leq 1$  gives  $\frac{B_1 x}{y} \leq A_1 \leq \frac{B_1 x}{y} + \frac{1}{y}$ , and so for fixed  $B_1$  the number of admissible  $A_1$ 's is  $\ll \frac{1}{y}$  and  $|A_1| \leq \frac{|B_1|+1}{y}$ . On the other hand, if  $B_1 \neq 0$ , then

$$|\Psi_{F;A_1,B_1,\alpha}(x, y)| \leq \frac{y^2}{\alpha^2 |B_1|^2},$$

providing, for every  $\delta \in (0, 1)$ ,

$$\begin{aligned} |I_{1,\delta,\alpha}(F)| &\ll \sum_{B_1 \in \mathbb{Z}^*} \int_0^1 \frac{y^2}{\alpha^2 |B_1|^2} \cdot \frac{1}{y} \cdot \frac{(|B_1| + 1)^\delta}{y^\delta} \int_0^y dx dy \\ &+ \sum_{A_1 \in \mathbb{N}} \iint_{0 \leq x \leq y \leq 1/A_1} dx dy \ll_\delta \alpha^{-2} + 1. \end{aligned}$$

Secondly, proceeding by induction on  $k$  allows us to reduce ourselves to the case when  $B_1 \neq 0$ ,  $B_k \neq 0$ , and  $A_i B_{i+1} - A_{i+1} B_i \neq 0$  for all  $1 \leq i < k$ . Indeed, notice that equality (4.5) shows that when there exists an  $i$  such that  $A_{i+1} B_i - A_i B_{i+1} = 0$ , the requirements  $(A_i, B_i) = (A_{i+1}, B_{i+1}) = 1$  lead to  $A_{i+1} = A_i$  and  $B_{i+1} = B_i$ , thus Proposition 6 follows from the situation where  $k$  is replaced by  $k - 1$  (see (4.2)). Similarly if  $B_1 = 0$  then  $\beta_{A_1,B_1,\alpha}(x, y) = 0$ , therefore

$$\beta_{A_1,B_1,\alpha}(x, y) - \beta_{A_2,B_2,\alpha}(x, y) = -\beta_{A_2,B_2,\alpha}(x, y).$$

Since  $F(0) = 1$  and  $F(x) = F(-x)$ , Proposition 6 once again reduces to the case of  $k - 1$  variables. The same argument allows us to assume that  $B_k \neq 0$ .

According to Lemma 8, and the previous remark, it will suffice to establish that the following expression converges for some  $\delta = \delta_\ell > 0$ ,

$$\begin{aligned} & \sum_{\substack{A_1, \dots, A_k, B_1, \dots, B_k \in \mathbb{Z} \\ A_i B_{i+1} - A_{i+1} B_i \neq 0 \\ (A_i, B_i) = 1, \forall i \\ B_1 \neq 0, B_k \neq 0}} \max_{i \in [1, k]} \{|A_i|^\delta, |B_i|^\delta\} \frac{1}{|B_1 B_k|} \prod_{i=1}^{k-1} \frac{1}{|A_i B_{i+1} - A_{i+1} B_i|} \\ & \times \iint_{0 \leq x \leq y \leq 1} y^2 \prod_{i=1}^k (A_i y - B_i x) \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy. \end{aligned} \quad (4.7)$$

Fix some  $\varepsilon \in (0, \frac{1}{1000k})$ . We start by making several reductions, the outcome of which is that we can focus on the scenario in which both of the following conditions hold:

- (I) The range of integration over  $y$  is restricted to  $y > \max_{i \in [1, k]} |B_i|^{-\varepsilon^2}$ .
- (II) For all  $i \in [1, k-1]$  we have  $|A_i B_{i+1} - A_{i+1} B_i| \ll \max_{i \in [1, k]} |B_i|^{\varepsilon^2}$ .

We then use two different arguments to handle the total contributions of the integers  $A_1, \dots, A_k, B_1, \dots, B_k$  for which there exists an index  $j \in [1, k-1]$  such that  $|B_{j+1}| > |B_j| \cdot \max_{i \in [1, k]} |B_i|^\varepsilon$ , and respectively the contributions of the integers for which there is no such index. Finally, we set  $\delta = \delta_\ell = \varepsilon^3$ .

In the remaining part of this section we will group the integers  $A_i$  and  $B_i$  into dy-adic ranges  $2^{a_i} - 1 \leq |A_i| \leq 2^{a_i+1}$  and  $2^{b_i} - 1 \leq |B_i| \leq 2^{b_i+1}$ , with  $a_i$  and  $b_i$  running through the non-negative integers. Note that the intervals  $[2^a - 1, 2^{a+1}]$  are overlapping, but this is not a problem because in this section we only add or integrate non-negative quantities.

**4.1. Disposing of the  $y$ 's for which  $y < \max_{i \in [1, k]} |B_i|^{-\varepsilon^2}$ .** With the grouping described above we can re-phrase the condition  $y < \max_{i \in [1, k]} |B_i|^{-\varepsilon^2}$  as  $y \ll 2^{-\varepsilon^2 \max(b_i)}$ . Notice also that

$$\begin{aligned} (A_i y - B_i x) \mathbf{1}_{0 < A_i y - B_i x \leq 1} & \leq 2 \min \{ \max\{|A_i|y, |B_i|x\}, 1 \} \\ & \ll \min \{ \max\{2^{a_i}y, 2^{b_i}x\}, 1 \} \end{aligned} \quad (4.8)$$

and

$$\min \{ (1/y)2^{-a_i} + 1, (1/x)2^{-b_i} + 1 \} \min \{ \max\{2^{a_i}y, 2^{b_i}x\}, 1 \} \leq 1. \quad (4.9)$$

The conditions  $0 < A_i y - B_i x \leq 1$  and  $0 \leq x \leq y \leq 1$  imply that  $|A_i| \leq |B_i| + \frac{1}{y}$ , so that if we assign  $A_i B_{i+1} - A_{i+1} B_i = D_i$ , then we have

$$\prod_{i=1}^{k-1} |D_i| \leq 2^{2(b_1 + \dots + b_k)} (1 + 1/y)^k =: L(\mathbf{b}, y).$$

Therefore, using also (4.8), and  $\max_i \{|A_i|^\delta, |B_i|^\delta\} \leq 2^{\varepsilon^3 \max(a_i)} 2^{\varepsilon^3 \max(b_i)}$ , we see that the expression in (4.7) is

$$\begin{aligned} & \ll \sum_{b_1, \dots, b_k \geq 0} \frac{2^{\varepsilon^3 \max(b_i)}}{2^{b_1 + b_k}} \int_0^{2^{-\varepsilon^2 \max(b_i)}} y^2 \sum_{1 \leq |D_1 \cdots D_{k-1}| \leq L(\mathbf{b}, y)} \frac{1}{|D_1| \cdots |D_{k-1}|} \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} \\ & \times \int_0^y \sum_{\mathbf{A}, \mathbf{B} \in \mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} \prod_{i=1}^{k-1} \min \{ \max \{ 2^{a_i} y, 2^{b_i} x \}, 1 \} \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy. \end{aligned} \quad (4.10)$$

According to Lemma 9 and (4.9) the expression after the innermost integral is

$$\ll \frac{2^{b_1}}{y} \prod_{i=1}^{k-1} \min \{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \} \min \{ \max \{ 2^{a_i} y, 2^{b_i} x \}, 1 \} \ll \frac{2^{b_1}}{y}.$$

Moreover, uniformly in  $y \in (0, 1]$ ,

$$\begin{aligned} & \sum_{1 \leq |D_1 \cdots D_{k-1}| \leq L(\mathbf{b}, y)} \frac{1}{|D_1| \cdots |D_{k-1}|} \ll (\log L(\mathbf{b}, y))^k \\ & \ll_k \left( b_1 + \cdots + b_k + \log(1 + 1/y) \right)^k \ll_k (b_1 + \cdots + b_k)^k \max_{j \in [0, k]} (\log(1 + 1/y))^j \end{aligned}$$

and

$$\sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 1 \ll \prod_{i=1}^k \log(2^{b_i+1} + 1/y) \ll b_1 \cdots b_k (\log(1 + 1/y))^k,$$

so

$$\begin{aligned} & \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} \ll \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} (2 \cdot 2^{\max(b_i)} + 1/y)^{\varepsilon^3} \\ & \ll \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(b_i)} y^{-\varepsilon^3} \ll b_1 \cdots b_k 2^{\varepsilon^3 \max(b_i)} y^{-\varepsilon^3} (\log(1 + 1/y))^k. \end{aligned}$$

It follows that the whole expression from (4.10) is bounded by

$$\begin{aligned}
& \sum_{b_1, \dots, b_k \geq 0} \frac{2^{2\varepsilon^3 \max(b_i)} (b_1 + \dots + b_k)^k}{2^{b_1 + b_k}} 2^{b_1} \int_0^{2^{-\varepsilon^2 \max(b_i)}} y^{1-\varepsilon^2} \max_{j \in [0, k]} (\log(1 + 1/y))^{j+k} dy \\
&= \sum_{b_1, \dots, b_k \geq 0} 2^{2\varepsilon^3 \max(b_i)} (b_1 + \dots + b_k)^k \int_{2^{\varepsilon^2 \max(b_i)}}^\infty \frac{\max_{j \in [k, 2k]} (\log(1 + u))^j}{u^{3-\varepsilon^2}} du \\
&\ll \sum_{b_1, \dots, b_k \geq 0} \frac{(b_1 + \dots + b_k)^k}{2^{(\varepsilon^2 - 2\varepsilon^3) \max(b_i)}} \ll_k \left( \sum_{b=1}^\infty \frac{b^{2k}}{2^{(\varepsilon^2 - 2\varepsilon^3)b/k}} \right)^k \ll_k 1.
\end{aligned}$$

**4.2. Disposing of  $k$ -tuples of integers with  $|A_j B_{j+1} - A_{j+1} B_j| > \max_{i \in [1, k]} |B_i|^{\varepsilon^2}$  for some  $j \in [1, k-1]$ .** By the union bound and the bound provided by Lemma 8 the contribution of such integers is

$$\begin{aligned}
& \ll \sum_{j=1}^k \sum_{\substack{A_1, \dots, A_k, B_1, \dots, B_k \in \mathbb{Z} \\ A_i B_{i+1} - A_{i+1} B_i \neq 0 \\ (A_i, B_i) = 1, \forall i}} \frac{\max_{i \in [1, k]} \{|A_i|^{\varepsilon^3}, |B_i|^{\varepsilon^3}\}}{|B_1 B_k|} \cdot \frac{1}{\max_{i \in [1, k]} |B_i|^{\varepsilon^2}} \prod_{i \neq j} \frac{1}{|A_i B_{i+1} - A_{i+1} B_i|} \\
& \quad \times \iint_{0 \leq x \leq y \leq 1} y^2 \prod_{i=1}^k (A_i y - B_i x) \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy.
\end{aligned}$$

It is enough to show that each of the inner expressions is convergent. We fix, for all  $i \neq j$ , values  $D_i = A_i B_{i+1} - A_{i+1} B_i$ . As before, we have  $\prod_{i \neq j} |D_i| \leq L(\mathbf{b}, y)$  with  $L(\mathbf{b}, y) = 2^{2(b_1 + \dots + b_k)} (1 + \frac{1}{y})^k$ . We are thus led to the following expression:

$$\begin{aligned}
& \sum_{b_1, \dots, b_k \geq 0} \frac{2^{(\varepsilon^3 - \varepsilon^2) \max(b_i)}}{2^{b_1 + b_k}} \int_0^1 y^2 \sum_{\substack{D_1, \dots, D_{j-1}, D_{j+1}, \dots, D_{\ell-1} \\ \prod_{i \neq j} |D_i| \leq L(\mathbf{b}, y)}} \prod_{\substack{1 \leq i \leq k-1 \\ i \neq j}} \frac{1}{|D_i|} \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} \\
& \times \int_0^y \sum_{\substack{A_1, \dots, A_k, B_1, \dots, B_k \in \mathbb{Z} \\ 2^{a_i} - 1 \leq |A_i| \leq 2^{a_i+1} \\ 2^{b_i} - 1 \leq |B_i| \leq 2^{b_i+1} \\ A_i B_{i+1} - A_{i+1} B_i = D_i, \forall i \neq j \\ (A_i, B_i) = 1, \forall i}} \prod_{i=1}^k \min \{ \max \{ 2^{a_i} y, 2^{b_i} x \}, 1 \} \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy. \quad (4.11)
\end{aligned}$$

We now apply Lemma 9 twice, first to the variable  $(A_i, B_i)$  with  $i \leq j$ , and then to the variables  $(A_\ell, B_\ell)$  with  $j+1 \leq \ell \leq k$  (in particular in the second application we reverse the order of the variables and identify  $A_{k-i}, B_{k-i}$  with  $A_{i+1}, B_{i+1}$  for



$i = 1, \dots, k - j - 1$ ). This gives,

$$\sum_{\substack{A_1, \dots, A_j, B_1, \dots, B_j \in \mathbb{Z} \\ 2^{a_i-1} \leq |A_i| \leq 2^{a_i+1}, \forall i \leq j \\ 2^{b_i-1} \leq |B_i| \leq 2^{b_i+1}, \forall i \leq j \\ A_i B_{i+1} - A_{i+1} B_i = D_i, \forall i < j \\ (A_i, B_i) = 1, \forall i \leq j}} \prod_{i=1}^j \mathbf{1}_{0 < A_i y - B_i x \leq 1} \ll \frac{2^{b_1}}{y} \prod_{i=1}^{j-1} \min \{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \},$$

$$\sum_{\substack{A_{j+1}, \dots, A_k, B_{j+1}, \dots, B_k \in \mathbb{Z} \\ 2^{a_i-1} \leq |A_i| \leq 2^{a_i+1}, \forall i \geq j+1 \\ 2^{b_i-1} \leq |B_i| \leq 2^{b_i+1}, \forall i \geq j+1 \\ A_i B_{i+1} - A_{i+1} B_i = D_i, \forall i \geq j+1 \\ (A_i, B_i) = 1, \forall i \geq j+1}} \prod_{i=j+1}^k \mathbf{1}_{0 < A_i y - B_i x \leq 1} \ll \frac{2^{b_k}}{y} \prod_{i=j+2}^k \min \{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \}.$$

In conjunction with (4.9) this shows that the expression inside the innermost integral in (4.11) is

$$\ll \frac{2^{b_1+b_k}}{y^2} \prod_{i \neq j, j+1} \min \{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \} \min \{ \max \{ 2^{a_i} y, 2^{b_i} x \}, 1 \} \ll \frac{2^{b_1+b_k}}{y^2}.$$

Using this bound and proceeding as in the previous case for the other sums, we conclude that (4.11) is

$$\ll \sum_{b_1, \dots, b_k \geq 0} 2^{(\varepsilon^3 - \varepsilon^2) \max(b_i)} \int_0^1 y \sum_{\substack{D_1, \dots, D_{j-1}, D_{j+1}, \dots, D_{k-1} \\ 1 \leq \prod_{i \neq j} |D_i| \leq L(\mathbf{b}, y)}} \prod_{\substack{1 \leq i \leq k-1 \\ i \neq j}} \frac{1}{|D_i|} \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i-1} \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} dy.$$

Using also  $2^{\varepsilon^3 \max(a_i)} \ll_{\varepsilon} 2^{\varepsilon^3 \max(b_i)} y^{-\varepsilon^3}$  and other estimates from the previous subsection we see that this is

$$\ll_{k, \varepsilon} \sum_{b_1, \dots, b_k \geq 0} \frac{(b_1 + \dots + b_k)^{2k}}{2^{(\varepsilon^2 - 2\varepsilon^3) \max(b_i)}} \int_1^\infty \frac{\max_{j \in [k, 2k]} (\log(1+u))^j}{u^{2-\varepsilon^3}} du \ll_k 1. \quad (4.12)$$

**4.3. (II) is fulfilled and  $b_{j+1} - b_j \leq \varepsilon \max_{i \in [1, k]} b_i$  for all  $j \in [1, k-1]$ .** Since (II) is fulfilled we have  $|D_i| \leq 2^{\varepsilon \max_{i \in [1, k]} b_i}$  for all  $i \in [1, k-1]$ , so it suffices to bound

above the expression

$$\begin{aligned}
& \sum_{b_1, \dots, b_k \geq 0} \frac{2^{\varepsilon^3 \max(b_i)}}{2^{b_1 + b_k}} \sum_{\substack{D_1, \dots, D_{k-1} \\ 1 \leq |D_i| \leq 2^{\varepsilon \max(b_i)}}} \frac{1}{|D_1| \cdots |D_{k-1}|} \\
& \times \int_0^1 y^2 \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} \int_0^y \sum_{(\mathbf{A}, \mathbf{B}) \in S_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} \prod_{i=1}^k \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy.
\end{aligned} \tag{4.13}$$

Next notice that  $0 < A_i y - B_i x \leq 1$  implies that  $|x - \frac{A_i}{B_i} y| \leq \frac{1}{|B_i|}$ . Therefore the contribution of the integral over  $x$  is  $\ll \min_{i \in [1, k]} \frac{1}{|B_i|} \ll 2^{-\max_{i \in [1, k]} b_i}$ .

Using that  $b_{j+1} - b_j \leq \varepsilon \max_{i \in [1, k]} b_i$  for all  $j \leq k-1$  and Lemma 9, we infer

$$\sum_{(\mathbf{A}, \mathbf{B}) \in S_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} 1 \ll 2^{a_1} 2^{b_1} \prod_{i=1}^{k-1} (2^{b_{i+1} - b_i} + 1) \ll 2^{a_1 + b_1} \cdot 2^{k\varepsilon \max(b_i)}. \tag{4.14}$$

It further follows from (4.14) that the expression in (4.13) is

$$\begin{aligned}
& \ll \sum_{b_1, \dots, b_k \geq 0} \frac{2^{\varepsilon^3 \max(b_i)}}{2^{b_1 + b_k}} \sum_{\substack{D_1, \dots, D_{k-1} \\ 1 \leq |D_i| \leq 2^{\varepsilon \max(b_i)}}} \frac{1}{|D_1| \cdots |D_{k-1}|} \\
& \times \int_0^1 y^2 \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i} - 1 \leq 2^{b_i+1} + 1/y}} 2^{b_1} \cdot 2^{\varepsilon^3 \max(a_i)} \cdot 2^{k\varepsilon \max(b_i)} \cdot 2^{-\max(b_i)} dy.
\end{aligned}$$

Proceeding as in the previous sections to handle the sums over  $a_i$  and the integral, we conclude that this is bounded above by the quantity in (4.12).

**4.4. (I) and (II) are fulfilled and there exists an index  $j \in [1, k-1]$  such that  $b_{j+1} - b_j > \varepsilon \max_{i \in [1, k]} b_i$ .** In this case because of (I) the range of integration is restricted to  $y > 2^{-\varepsilon^2 \max_{i \in [1, k]} b_i}$  and we take  $|D_i| \leq 2^{\varepsilon^2 \max_{i \in [1, k]} b_i}$  due to (II). Note that the  $\varepsilon^2$  in the bound for  $|D_i|$  is important because will be matched against the larger  $\varepsilon$  in  $b_{j+1} - b_j \geq \varepsilon \max_{i \in [1, k]} b_i$  at a crucial point in the argument. Therefore in

this case it is enough to bound

$$\begin{aligned}
& \sum_{\substack{b_1, \dots, b_k \geq 0 \\ \exists j, b_{j+1} - b_j \geq \varepsilon \max(b_i)}} \frac{2^{\varepsilon^3 \max(b_i)}}{2^{b_1 + b_k}} \sum_{\substack{D_1, \dots, D_{k-1} \\ 1 \leq |D_i| \leq 2^{\varepsilon^2 \max(b_i)}}} \frac{1}{|D_1| \cdots |D_{k-1}|} \int_{2^{-\varepsilon^2 \max(b_i)}}^1 y^2 \\
& \times \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i-1} \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} \int_0^y \sum_{(\mathbf{A}, \mathbf{B}) \in \mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} \prod_{i=1}^k (A_i y - B_i x) \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy.
\end{aligned} \tag{4.15}$$

Using the union bound we fix an index  $j$  such that  $b_{j+1} - b_j \geq \varepsilon \max(b_i)$ . In the integral above we are requiring  $0 < A_j y - B_j x \leq 1$  and  $0 < A_{j+1} y - B_{j+1} x \leq 1$ . Set

$$\xi_1 := A_j y - B_j x \in [0, 1], \quad \xi_2 = A_{j+1} y - B_{j+1} x \in [0, 1].$$

Solving this system of equations we see that

$$y = \frac{B_{j+1} \xi_1 - B_j \xi_2}{D_j}.$$

This leads to

$$B_{j+1} \xi_1 = O(y |D_j| + |B_j|) = O(2^{\varepsilon^2 \max(b_i)} + 2^{b_j}).$$

Using also  $b_{j+1} - b_j \geq \varepsilon \max(b_i)$  and  $b_j \geq 0$  we infer

$$\xi_1 \ll 2^{\varepsilon^2 \max(b_i) - b_{j+1}} + 2^{b_j - b_{j+1}} \leq 2^{(\varepsilon^2 - \varepsilon) \max(b_i)} + 2^{-\varepsilon \max(b_i)} \leq 2 \cdot 2^{-(\varepsilon/2) \max(b_i)}.$$

Therefore  $\xi_1 = |A_j y - B_j x| \ll 2^{(-\varepsilon/2) \max(b_i)}$ .

It follows therefore that the expression in (4.15) is

$$\begin{aligned}
& \ll \sum_{b_1, \dots, b_k \geq 0} \frac{2^{\varepsilon^3 \max(b_i)}}{2^{b_1 + b_k}} \sum_{\substack{D_1, \dots, D_{k-1} \\ 1 \leq |D_i| \leq 2^{\varepsilon^2 \max(b_i)}}} \frac{1}{|D_1| \cdots |D_{k-1}|} \int_{2^{-\varepsilon^2 \max(b_i)}}^1 y^2 \\
& \times \sum_{\substack{a_1, \dots, a_k \\ 2^{a_i-1} \leq 2^{b_i+1} + 1/y}} 2^{\varepsilon^3 \max(a_i)} \int_0^y 2^{-(\varepsilon/2) \max(b_i)} \sum_{(\mathbf{A}, \mathbf{B}) \in \mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} \prod_{i=1}^k \mathbf{1}_{0 < A_i y - B_i x \leq 1} dx dy.
\end{aligned} \tag{4.16}$$

Using Lemma 9 and our assumption that  $y > 2^{-\varepsilon^2 \max(b_i)}$ , we see that

$$\sum_{(\mathbf{A}, \mathbf{B}) \in \mathcal{S}_{\mathbf{D}, \mathbf{a}, \mathbf{b}}} \mathbf{1}_{0 < A_i y - B_i x \leq 1} \ll 2^{b_1} 2^{k \varepsilon^2 \max(b_i)}.$$

Combining everything together we conclude that the expression in (4.16) is

$$\ll_{k, \varepsilon} \sum_{b_1, \dots, b_k \geq 0} \frac{(b_1 + \cdots + b_k)^{2k}}{2^{(\varepsilon/2 - k \varepsilon^2 + 2 \varepsilon^3) \max(b_i)}} \ll_{k, \varepsilon} 1.$$

## 5. ASYMPTOTIC FORMULAS FOR THE MOMENTS OF THE LARGE SIEVE MATRIX

Fix a non-decreasing  $C^\infty$  function  $\Xi : \mathbb{R} \rightarrow [0, 1]$  with  $\Xi \equiv 0$  on  $(-\infty, 0]$ ,  $\Xi \equiv 1$  on  $[1, \infty)$ , and

$$\Xi^{(k)}(0) = \Xi^{(k)}(1) = 0, \quad \forall k \in \mathbb{N}, \quad \Xi(x) + \Xi(1-x) = 1, \quad \forall x \in [0, 1].$$

Fix  $c \in (0, 1)$  and let  $\delta = N^{-1+c} > 0$ . Consider the function  $f_\delta \in C_c^\infty(\mathbb{R})$  defined by  $f_\delta(x) \equiv 0$  on  $(-\infty, 0] \cup [1 + \delta, \infty)$ ,  $f_\delta(x) \equiv 1$  on  $[\delta, 1]$ ,  $f_\delta(x) = \Xi(\frac{x}{\delta})$  if  $x \in [0, \delta]$ , and  $f_\delta(x) = \Xi(\frac{1+\delta-x}{\delta})$  if  $x \in [1, 1 + \delta]$ . Consider also the function defined by

$$\phi(u) = \int_0^1 \Xi'(y) e(-uy) dy.$$

A direct calculation provides  $f_\delta(x) + f_\delta(x+1) = 1$  for all  $x \in [0, 1]$ , and

$$\widehat{f}_\delta(u) = \frac{1 - e(-u)}{2\pi i u} \phi(\delta u) = e^{-\pi i u} \operatorname{sinc}(\pi u) \phi(\delta u) = \widehat{\chi}_{[0,1]}(u) (1 - \phi(\delta u)).$$

It is clear that  $\|\phi\|_\infty \leq \phi(0) = 1$ ,  $\phi(u) = 1 + O(|u|)$ , and

$$\phi(u) = O_A(|u|^{-A}), \quad \forall A > 0.$$

Since  $|\operatorname{sinc}(\pi u)| \leq \frac{1}{|u|}$  and  $\|\widehat{f}_\delta\|_\infty \leq 1$ , we also infer

$$\|\widehat{f}_\delta - \widehat{\chi}_{[0,1]}\|_\infty = \sup_{u \in \mathbb{R} \setminus \{0\}} (|\operatorname{sinc}(\pi u)| \cdot |1 - \phi(\delta u)|) \ll \sup_{u \in \mathbb{R} \setminus \{0\}} ((1/|u|)\delta|u|) = \delta,$$

and taking  $\Psi_F$  as in (4.2),

$$\|\Psi_{\widehat{f}_\delta} - \Psi_{\widehat{\chi}_{[0,1]}}\|_\infty \ll \delta. \quad (5.1)$$

It is also plain that

$$\widehat{f}_\delta'(x) = -2\pi i \int_{\mathbb{R}} \xi f_\delta(\xi) e(-x\xi) d\xi = O(1).$$

Integrating by parts and employing  $\|f_\delta^{(A)}\|_\infty \ll_A \delta^{-A}$ , we obtain

$$\widehat{f}_\delta(x) = \frac{1}{(2\pi i x)^A} \int_{\mathbb{R}} f_\delta^{(A)}(\xi) e(-x\xi) d\xi \ll_A |x|^{-A} \delta \cdot \delta^{-A} = \frac{N^{(1-c)(A-1)}}{|x|^A}. \quad (5.2)$$

As shown in (3.10),  $\mathfrak{M}_Q(\ell)$  can be replaced by the smoothed sum  $\mathfrak{M}_{Q;\delta}(\ell)$  defined in (3.3). Consider the associated  $\mathbb{Z}$ -periodic function defined by

$$F_Q(x) := \sum_{k \in \mathbb{Z}} \widehat{f}_\delta(N(x+k)) = \sum_{k \in \mathbb{Z}} c_k e(kx). \quad (5.3)$$

Its Fourier coefficients,

$$c_n = \int_0^1 F_Q(x) e(-nx) dx = \int_{\mathbb{R}} \widehat{f}_\delta(Nu) e(-nu) du = \frac{1}{N} f_\delta\left(\frac{-n}{N}\right), \quad (5.4)$$

satisfy  $0 \leq c_n \leq \frac{1}{N}$  for all  $n$ ,  $c_n = 0$  unless  $-(1+\delta)N < n < 0$ , and

$$F_Q(0) = \sum_{k \in \mathbb{Z}} c_k = 1 + O(\delta). \quad (5.5)$$

We also have  $\|F_Q\|_\infty \leq 2$ .

With  $\mathbf{x} = (x_1, \dots, x_\ell)$ ,  $\xi = (\xi_1, \dots, \xi_\ell)$ , the Fourier transform of the function  $h_{\delta;\Theta}$  defined in (3.2) is given by

$$\begin{aligned} \widehat{h}_{\delta;\Theta}(\mathbf{x}) &= \int_{\mathbb{R}^\ell} e(-\mathbf{x} \cdot \xi) h_{\delta;\Theta}(\xi) d\xi \\ &= N^\ell \widehat{f}_\delta(N(x_1 + \theta_\ell - \theta_1)) \widehat{f}_\delta(N(x_2 + \theta_1 - \theta_2)) \cdots \widehat{f}_\delta(N(x_\ell + \theta_{\ell-1} - \theta_\ell)). \end{aligned}$$

Poisson summation, (3.3), and the above formula for  $\widehat{h}_{\delta;\Theta}(n_1, \dots, n_\ell)$  provide

$$\mathfrak{M}_{Q;\delta}(\ell) = \frac{1}{N} \sum_{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q} F_Q(\theta_\ell - \theta_1) F_Q(\theta_1 - \theta_2) \cdots F_Q(\theta_{\ell-1} - \theta_\ell).$$

Taking

$$\begin{aligned} F_{Q;n}(x) &:= F_Q(x) e(-nx) = \sum_{k \in \mathbb{Z}} c_{n+k} e(kx), \\ S_{Q;n}(\ell) &:= \sum_{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q} F_{Q;n}(\theta_1 - \theta_2) F_{Q;n}(\theta_2 - \theta_3) \cdots F_{Q;n}(\theta_{\ell-1} - \theta_\ell), \end{aligned}$$

and employing (5.4) to express  $F_Q(\theta_\ell - \theta_1)$  we can write

$$\begin{aligned} \mathfrak{M}_{Q;\delta}(\ell) &= \frac{1}{N} \sum_{n \in \mathbb{Z}} c_n \sum_{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q} e(n(\theta_\ell - \theta_1)) \prod_{i=1}^{\ell-1} F_{Q;n}(\theta_i - \theta_{i+1}) \\ &= \frac{1}{N} \sum_{-(1+\delta)N < n < 0} c_n S_{Q;n}(\ell). \end{aligned} \quad (5.6)$$

Next we focus on  $S_{Q;n}(\ell)$ , which is expressed after replacing  $\theta_\ell$  by  $-\theta_\ell$ , as

$$\begin{aligned} &\sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ n_1, \dots, n_{\ell-1} \in \mathbb{Z}}} c_{n+n_1} \cdots c_{n+n_{\ell-1}} e(n_1(\theta_1 - \theta_2) + n_2(\theta_2 - \theta_3) + \cdots + n_{\ell-1}(\theta_{\ell-1} - \theta_\ell)) \\ &= \sum_{\substack{\theta_1, \dots, \theta_\ell \in \mathcal{F}_Q \\ n_1, \dots, n_{\ell-1} \in \mathbb{Z}}} c_{n+n_1} \cdots c_{n+n_{\ell-1}} e(n_1\theta_1 + (n_2 - n_1)\theta_2 + \cdots + (n_{\ell-1} - n_{\ell-2})\theta_{\ell-1} + n_{\ell-1}\theta_\ell). \end{aligned}$$

Upon (2.5), with  $\mathbf{r} := (r_1, \dots, r_{\ell-1})$ ,  $\mathbf{d} := (d_1, \dots, d_{\ell-1})$ , this can also be written as

$$\begin{aligned}
S_{Q;n}(\ell) &= \sum_{n_1, \dots, n_{\ell-1} \in \mathbb{Z}} c_{n+n_1} \cdots c_{n+n_{\ell-1}} \sum_{\substack{d_1 | n_1 \\ d_2 | n_2 - n_1 \\ \vdots \\ d_{\ell-1} | n_{\ell-1} - n_{\ell-2} \\ d_\ell | n_{\ell-1}}} d_1 \cdots d_\ell M\left(\frac{Q}{d_1}\right) \cdots M\left(\frac{Q}{d_\ell}\right) \\
&= \sum_{\mathbf{d} \in [1, Q]^{\ell-1}} M\left(\frac{Q}{d_1}\right) \cdots M\left(\frac{Q}{d_{\ell-1}}\right) \sum_{\mathbf{r} \in \mathbb{Z}^{\ell-1}} d_1 \cdots d_{\ell-1} \\
&\quad \times c_{n+d_1 r_1} c_{n+d_1 r_1 + d_2 r_2} \cdots c_{n+d_1 r_1 + \cdots + d_{\ell-1} r_{\ell-1}} \sum_{d_\ell | d_1 r_1 + \cdots + d_{\ell-1} r_{\ell-1}} d_\ell M\left(\frac{Q}{d_\ell}\right).
\end{aligned} \tag{5.7}$$

Taking into account (2.5) and (5.4) we can express the inner two sums in (5.7) as

$$\begin{aligned}
&\sum_{\substack{\mathbf{r} \in \mathbb{Z}^{\ell-1} \\ \theta \in \mathcal{F}_Q}} c_{n+d_1 r_1} c_{n+d_1 r_1 + d_2 r_2} \cdots c_{n+d_1 r_1 + \cdots + d_{\ell-1} r_{\ell-1}} e\left(-\theta \sum_{j=1}^{\ell-1} d_j r_j\right) \\
&= \sum_{\substack{\mathbf{r} \in \mathbb{Z}^{\ell-1} \\ \theta \in \mathcal{F}_Q}} e\left(-\theta \sum_{j=1}^{\ell-1} d_j r_j\right) \int_{\mathbb{R}^{\ell-1}} \prod_{k=1}^{\ell-1} \widehat{f}_\delta(Nu_k) e\left(-\sum_{j=1}^{\ell-1} \left(n + \sum_{i=1}^j d_i r_i\right) u_j\right) d\mathbf{u} \\
&= \sum_{\substack{\mathbf{r} \in \mathbb{Z}^{\ell-1} \\ \theta \in \mathcal{F}_Q}} \int_{\mathbb{R}^{\ell-1}} e\left(-\sum_{j=1}^{\ell-1} d_j r_j (u_j + \cdots + u_{\ell-1} + \theta) - n \sum_{j=1}^{\ell-1} u_j\right) \prod_{k=1}^{\ell-1} \widehat{f}_\delta(Nu_k) d\mathbf{u} \\
&= \sum_{\substack{\mathbf{r} \in \mathbb{Z}^{\ell-1} \\ \theta \in \mathcal{F}_Q}} \int_{\mathbb{R}^{\ell-1}} e\left(-\sum_{j=1}^{\ell-1} d_j r_j \sum_{i=j}^{\ell-1} u_i - n \sum_{j=1}^{\ell-1} u_j + n\theta\right) \widehat{f}_\delta(N(u_{\ell-1} - \theta)) \prod_{k=1}^{\ell-2} \widehat{f}_\delta(Nu_k) d\mathbf{u}.
\end{aligned}$$

With  $\mathbf{y} = (y_1, \dots, y_{\ell-1})$ , denote

$$H_{Q;\mathbf{d},\theta,n}(\mathbf{y}) := e\left(n\left(\theta - \frac{y_1}{d_1}\right)\right) \prod_{k=1}^{\ell-2} \widehat{f}_\delta\left(N\left(\frac{y_k}{d_k} - \frac{y_{k+1}}{d_{k+1}}\right)\right) \widehat{f}_\delta\left(N\left(\frac{y_{\ell-1}}{d_{\ell-1}} - \theta\right)\right).$$

The change of variables  $y_i = d_i(u_i + \cdots + u_{\ell-1})$ ,  $i = 1, \dots, \ell-1$ , provides

$$\begin{aligned}
u_1 &= \frac{y_1}{d_1} - \frac{y_2}{d_2}, \quad u_2 = \frac{y_2}{d_2} - \frac{y_3}{d_3}, \quad \dots, \quad u_{\ell-2} = \frac{y_{\ell-2}}{d_{\ell-2}} - \frac{y_{\ell-1}}{d_{\ell-1}}, \quad u_{\ell-1} = \frac{y_{\ell-1}}{d_{\ell-1}}, \\
&- \sum_{j=1}^{\ell-1} d_j r_j \sum_{i=j}^{\ell-1} u_i - n \sum_{j=1}^{\ell-1} u_j + n\theta = -\mathbf{r} \cdot \mathbf{y} + n\left(\theta - \frac{y_1}{d_1}\right),
\end{aligned}$$

and the contribution of the two inner sums in (5.7) becomes

$$\begin{aligned} \sum_{\theta \in \mathcal{F}_Q} \sum_{\mathbf{r} \in \mathbb{Z}^{\ell-1}} \int_{\mathbb{R}^{\ell-1}} e(-\mathbf{r} \cdot \mathbf{y}) H_{Q;\mathbf{d},\theta,n}(\mathbf{y}) d\mathbf{y} &= \sum_{\theta \in \mathcal{F}_Q} \sum_{\mathbf{r} \in \mathbb{Z}^{\ell-1}} \widehat{H}_{Q;\mathbf{d},\theta,n}(\mathbf{r}) \\ &= \sum_{\theta \in \mathcal{F}_Q} \sum_{\mathbf{r} \in \mathbb{Z}^{\ell-1}} H_{Q;\mathbf{d},\theta,n}(\mathbf{r}), \end{aligned}$$

where Poisson summation was used in the last equality. Inserting this back into (5.7) and setting  $\mu(\mathbf{k}) := \mu(k_1) \cdots \mu(k_{\ell-1})$ , we find

$$\begin{aligned} S_{Q;n}(\ell) &= \sum_{\mathbf{d} \in [1,Q]^{\ell-1}} M\left(\frac{Q}{d_1}\right) \cdots M\left(\frac{Q}{d_{\ell-1}}\right) \sum_{\theta \in \mathcal{F}_Q} \sum_{\mathbf{r} \in \mathbb{Z}^{\ell-1}} H_{Q;\mathbf{d},\theta,n}(\mathbf{r}) \\ &= \sum_{\mathbf{k} \in [1,Q]^{\ell-1}} \mu(\mathbf{k}) \sum_{\mathbf{r} \in \mathbb{Z}^{\ell-1}} \sum_{d_i \in [1, \frac{Q}{k_i}], \forall i} \sum_{\theta \in \mathcal{F}_Q} H_{Q;\mathbf{d},\theta,n}(\mathbf{r}). \end{aligned} \tag{5.8}$$

Consider the functions

$$\begin{aligned} \widetilde{F}_{Q;n}(x_1, \dots, x_{\ell-1}) &:= e\left(-\frac{nx_1}{N}\right) \prod_{i=1}^{\ell-2} \widehat{f}_{\delta}(x_i - x_{i+1}) \widehat{f}_{\delta}(x_{\ell-1}), \\ h_{Q;e_i,\Delta_i}(b, q) &:= \beta_{e_i,\Delta_i,N}(b, q) = \frac{N\Delta_i}{q(e_i q - \Delta_i b)}, \\ G_{Q;\mathbf{e},\Delta} &:= \sum_{n \in \mathbb{Z}} c_n \widetilde{F}_{Q;n}(h_{Q;e_1,\Delta_1}, \dots, h_{Q;e_{\ell-1},\Delta_{\ell-1}}) \\ &= F_Q\left(-\frac{1}{N} h_{Q;e_1,\Delta_1}\right) \prod_{i=1}^{\ell-2} \widehat{f}_{\delta}(h_{Q;e_i,\Delta_i} - h_{Q;e_{i+1},\Delta_{i+1}}) \widehat{f}_{\delta}(h_{Q;e_{\ell-1},\Delta_{\ell-1}}), \\ \Phi_{Q;\mathbf{e},\Delta} &:= \Psi_{\widehat{f}_{\delta}}(h_{Q;e_1,\Delta_1}, \dots, h_{Q;e_{\ell-1},\Delta_{\ell-1}}). \end{aligned}$$

Denote  $\theta = \frac{a}{q} \in \mathcal{F}_Q$  and let  $b = \bar{a} \in [1, q]$  such that  $a\bar{a} \equiv 1 \pmod{q}$ . Setting  $\Delta_i := r_i q - d_i a$ , we have  $d_i \equiv -\Delta_i b \pmod{q}$ ,  $e_i := \frac{d_i + \Delta_i b}{q} \in \mathbb{Z}$ , and

$$1 \leq d_i = e_i q - \Delta_i b \leq \frac{Q}{k_i}.$$

Note that if  $\Delta_i = 0$  for some  $i$ , then  $e_i \geq 1$ . Employing also

$$\frac{r_i}{d_i} - \frac{r_{i+1}}{d_{i+1}} = \left(\frac{r_i}{d_i} - \frac{a}{q}\right) - \left(\frac{r_{i+1}}{d_{i+1}} - \frac{a}{q}\right) = \frac{\Delta_i}{qd_i} - \frac{\Delta_{i+1}}{qd_{i+1}},$$

we can rewrite

$$H_{Q;\mathbf{d},\theta,n}(\mathbf{r}) = \widetilde{F}_{Q;n}\left(\frac{N\Delta_1}{qd_1}, \dots, \frac{N\Delta_{\ell-1}}{qd_{\ell-1}}\right).$$

Subsequently, with  $\Delta = (\Delta_1, \dots, \Delta_{\ell-1})$ ,  $\mathbf{e} = (e_1, \dots, e_{\ell-1})$ , using the second expression in (5.6) for  $\mathfrak{M}_{Q;\delta}(\ell)$ , equality (5.8) and the formulas for  $H_{Q;\mathbf{d},\theta,n}$  and  $G_{Q;\mathbf{e},\Delta}$ , we infer

$$\mathfrak{M}_{Q;\delta}(\ell) = \frac{1}{N} \sum_{\mathbf{k} \in [1,Q]^{\ell-1}} \mu(\mathbf{k}) \sum_{\mathbf{e}, \Delta \in \mathbb{Z}^{\ell-1}} \sum_{\substack{1 \leq b \leq q \leq Q, (q,b)=1 \\ 1 \leq d_i := e_i q - \Delta_i, b \leq \frac{Q}{k_i}, \forall i}} G_{Q;\mathbf{e},\Delta}(b, q). \quad (5.9)$$

**Lemma 10.** *If  $|\beta| \leq \frac{N}{2}$ , then for every integer  $B > 0$ :*

$$F_Q\left(\frac{\beta}{N}\right) = \sum_{n \in \mathbb{Z}} \widehat{f}_{\delta}(\beta + Nn) = \widehat{f}_{\delta}(\beta) + O_B(N^{-B}).$$

*Proof.* An application of (5.2) gives, for every  $A > 0$ ,

$$\left| \sum_{n \in \mathbb{Z}} \widehat{f}_{\delta}(\beta + Nn) - \widehat{f}_{\delta}(\beta) \right| \ll_A \sum_{|n| \geq 1} \frac{N^{(1-c)(A-1)}}{|\beta + Nn|^A} \ll_A N^{(1-c)(A-1)-A}.$$

The desired estimate follows choosing  $A$  with  $A - (1-c)(A-1) = cA + 1 - c > B$ .  $\square$

Since  $c_m = 0$  for  $|m| > 2N$ , the definitions of  $d_i, \Delta_i, e_i$  and the condition

$$c_n c_{n+d_1 r_1} \cdots c_{n+d_1 r_1 + \dots + d_{\ell} r_{\ell}} \neq 0$$

trivially imply  $d_i \leq Q$ ,  $|r_i| \ll N$ ,  $|\Delta_i| \ll QN$ . Since  $\Delta_i \equiv -d_i a \pmod{q}$ , for fixed  $d_i$  and  $q$  the number of admissible values for  $\Delta_i$  is  $\ll \frac{NQ}{q}$ . This provides

$$\begin{aligned} \#\text{non-zero terms in (5.9)} &\ll \sum_{\mathbf{k} \in [1,Q]^{\ell-1}} \frac{Q^{\ell-1}}{k_1 \cdots k_{\ell-1}} \sum_{q \in [1,Q]} q \left( \frac{NQ}{q} \right)^{\ell-1} \\ &\ll N^{2\ell-2} (\log Q)^{\ell}. \end{aligned} \quad (5.10)$$

Let  $\Lambda = N^d$  with  $\boxed{0 < 1 - c < d}$ , where we think of  $d > 0$  small (to be indicated precisely later) and of  $c$  as being close to 1. Denote  $\beta_i := h_{Q;e_i,\Delta_i}(b, q) = \frac{N\Delta_i}{qd_i}$ .

**Lemma 11.** *The contribution to  $\mathfrak{M}_{Q;\delta}(\ell)$  in (5.9) from terms with  $|\beta_i - \beta_{i+1}| > \Lambda$  for some  $i \in [1, \ell-2]$ ,  $|\beta_{\ell-1}| > \Lambda$ , or  $|\beta_1| > \Lambda$  is  $\ll_{\ell,B} N^{-B}$  for every  $B > 0$ .*

*Proof.* In (5.2) we choose  $A > 0$  such that  $\boxed{(c+d-1)A > B + 2\ell}$ . In the first two cases (5.2) provides  $|\widehat{f}_{\delta}(\beta_i - \beta_{i+1})| \ll_A N^{(1-c)(A-1)-dA} = N^{(1-c-d)A+c-1}$ , and respectively  $|\widehat{f}_{\delta}(\beta_{\ell-1})| \ll_A N^{(1-c-d)A+c-1}$ . Combining this with (5.9), (5.10),  $\|\widehat{f}_{\delta}\|_{\infty} \leq 1$  and  $\|F_{\delta,n}\|_{\infty} \leq 2$ , we infer that the contribution of these two cases to  $\mathfrak{M}_{Q;\delta}(\ell)$  is  $\ll_{\ell,A} N^{-1} N^{2\ell-1} N^{(1-c-d)A+c-1} \ll_B N^{-B}$ . If this is not the case, then  $|\beta_{\ell-1}| \leq \Lambda$  and  $|\beta_i - \beta_{i+1}| \leq \Lambda$  for every  $i \in [1, \ell-2]$ , and so necessarily  $|\beta_1| \leq (\ell-1)\Lambda < \frac{N}{2}$ .



Applying Lemma 10 and proceeding as above with  $\Lambda < |\beta_1| \leq \frac{N}{2}$  we conclude that the contribution of the case  $|\beta_1| > \Lambda$  to  $\mathfrak{M}_{Q;\delta}(\ell)$  is again  $\ll_B N^{-B}$ .  $\square$

We now work only with  $\max\{|\beta_1|, \dots, |\beta_{\ell-1}|\} \leq \Lambda$  and denote the resulting contributions to  $\mathfrak{M}_{Q;\delta}(\ell)$  by  $\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell)$ . Let us remark first that  $|\beta_i| = \frac{N|\Delta_i|}{qd_i} \ll \Lambda$  yields  $|\Delta_i| \ll \frac{\Lambda q d_i}{N} \ll \frac{\Lambda q}{Q^2} \cdot \frac{Q}{k_i} \ll \frac{\Lambda}{k_i}$ . We also have  $d_i \leq \frac{Q}{k_i}$ , hence  $\frac{N|\Delta_i|}{q} \ll \Lambda d_i \ll \frac{\Lambda Q}{k_i}$ , leading to  $k_i \ll \frac{\Lambda Q q}{N|\Delta_i|} \ll \frac{\Lambda}{|\Delta_i|} \leq \Lambda$ . Notice also that  $\min\{\frac{Q}{d_i}, \frac{Q|\Delta_i|}{q}\} \geq 1$  and  $\frac{Q}{d_i} \cdot \frac{Q|\Delta_i|}{q} \ll \frac{N|\Delta_i|}{qd_i} \ll \Lambda$ , and thus  $\frac{Q}{d_i} \ll \Lambda$  and  $\frac{Q}{\Lambda} \leq \frac{Q|\Delta_i|}{\Lambda} \ll q$ , showing also that  $|e_i| \ll \frac{Q}{q} + |\Delta_i| \ll \Lambda$ .

If the region

$$\Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q) := \left\{ (b, q) \in [0, Q]^2 : b \leq q, \frac{N|\Delta_i|}{\Lambda q} < e_i q - \Delta_i b \leq \frac{Q}{k_i}, \forall i \in [1, \ell-1] \right\}$$

is nonempty, then  $k_i, |\Delta_i|, |e_i| \ll \Lambda$  (as above) together with Lemma 10 show that the price of replacing  $\sum_{n \in \mathbb{Z}} c_n e(-\frac{n\beta_1}{N})$  by  $\widehat{f}_\delta(-\beta_1)$  in  $\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell)$  as in (5.9) is  $\ll_B \frac{1}{N} \Lambda^{3\ell} Q^2 N^{-B}$ . Accordingly, for every  $B > 0$ :

$$\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) = \frac{1}{N} \sum_{\mathbf{k} \in [1, Q]^{\ell-1}} \mu(\mathbf{k}) \sum_{\mathbf{e}, \Delta \in \mathbb{Z}^{\ell-1}} \sum_{(b, q) \in \Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q) \cap \mathbb{Z}^{\ell-1} \atop (q, b)=1} \Phi_{Q; \mathbf{e}, \Delta}(b, q) + O_B(N^{-B}).$$

We wish to apply Lemma 3 to the function  $f = \Phi_{Q; \mathbf{e}, \Delta}$  in the region  $\Omega = \Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q)$ . Denote by  $\|\cdot\|_\infty$  the sup norm on  $\Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q)$ . It is plain that

$$\begin{aligned} \left\| \frac{\partial \Phi_{Q; \mathbf{e}, \Delta}}{\partial b} \right\|_\infty &= N \Delta_i^2 \left\| \frac{1}{q(e_i q - \Delta_i b)^2} \right\|_\infty \ll N \Delta_i^2 \frac{\Lambda^2 Q}{N^2 \Delta_i^2} \ll \frac{\Lambda^2 Q}{N} \ll \frac{\Lambda^2}{Q}, \\ \left\| \frac{\partial \Phi_{Q; \mathbf{e}, \Delta}}{\partial q} \right\|_\infty &= N |\Delta_i| \left\| \frac{2e_i q - \Delta_i b}{q^2(e_i q - \Delta_i b)^2} \right\|_\infty \ll N |\Delta_i| \frac{(q+b)\Lambda}{q^2} \cdot \frac{\Lambda^2 q^2}{N^2 \Delta_i^2} \ll \frac{\Lambda^3 Q}{N} \ll \frac{\Lambda^3}{Q}, \\ \|\Phi_{Q; \mathbf{e}, \Delta}\|_\infty &\ll_\ell 1, \quad \|D\Phi_{Q; \mathbf{e}, \Delta}\|_\infty \leq \left\| \frac{\partial \Phi_{Q; \mathbf{e}, \Delta}}{\partial b} \right\|_\infty + \left\| \frac{\partial \Phi_{Q; \mathbf{e}, \Delta}}{\partial q} \right\|_\infty \ll_\ell \frac{\Lambda^3}{Q}. \end{aligned}$$

The boundary of  $\Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q)$  is the union of at most  $2\ell + 1$  line segments and parabola arcs, so Lemma 3 applies and yields

$$\sum_{(b, q) \in \Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q) \cap \mathbb{Z}^2 \atop (q, b)=1} \Phi_{Q; \mathbf{e}, \Delta}(b, q) = \frac{6}{\pi^2} \iint_{\Omega_{\mathbf{e}, \Delta, \mathbf{k}}(Q)} \Phi_{Q; \mathbf{e}, \Delta}(x, y) dx dy + \mathcal{E}_{\mathbf{e}, \Delta, \mathbf{k}}(Q),$$

with error  $\mathcal{E}_{\mathbf{e}, \Delta, \mathbf{k}}(Q) \ll_\ell \Lambda^3 Q \log Q$ . Due to the constraints  $\max\{k_i, |e_i|\} \ll \Lambda$  and  $|\Delta_i| \ll \frac{\Lambda}{k_i}$ , this contributes to  $\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell)$  by a quantity that is

$$\ll_\ell N^{-1} (\Lambda^2 \log \Lambda)^{\ell-1} \Lambda^3 Q \log Q \ll Q^{-\alpha_\ell} (\log Q)^\ell,$$

where  $\boxed{0 < \alpha_\ell := 1 - (4\ell + 2)d < 1}$ .

Rescaling to  $(b, q) = (Qx, Qy)$ , we find

$$\begin{aligned} \mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) &= \frac{6Q^2}{\pi^2 N} \sum_{\mathbf{k} \in [1, Q]^{\ell-1}} \mu(\mathbf{k}) \sum_{\mathbf{e}, \Delta \in \mathbb{Z}^{\ell-1}} \iint_{\tilde{\Omega}_{\mathbf{e}, \Delta, \mathbf{k}}(Q)} \Phi_{Q; \mathbf{e}, \Delta}(x, y) dx dy \\ &\quad + O_\ell(Q^{-\alpha_\ell}(\log Q)^\ell), \end{aligned} \quad (5.11)$$

where

$$\tilde{\Omega}_{\mathbf{e}, \Delta, \mathbf{k}}(Q) := \left\{ (x, y) \in [0, 1]^2 : x \leq y, \frac{N|\Delta_i|}{\Lambda Q^2 y} < e_i y - \Delta_i x \leq \frac{1}{k_i}, \forall i \in [1, \ell-1] \right\}.$$

Note that  $\tilde{\Omega}_{\mathbf{e}, \Delta, \mathbf{k}}(Q) \neq \emptyset$  produces  $k_i, |\Delta_i|, |e_i| \ll \Lambda$ . This is because  $|\Delta_i| \leq \frac{\Lambda Q^2 y}{N} \ll \Lambda$ ,  $k_i \leq \frac{1}{e_i y - \Delta_i x} \leq \frac{\Lambda Q^2 y}{N} \ll \Lambda$ ,  $\frac{1}{y} \leq \frac{\Lambda Q^2}{N k_i} \ll \Lambda$ , and  $|e_i| \leq \frac{|e_i y - \Delta_i x|}{y} + \frac{|\Delta_i| y}{y} \leq \frac{1}{y} + |\Delta_i| \ll \Lambda$ .

If  $0 < e_i y - \Delta_i x \leq \frac{N|\Delta_i|}{\Lambda Q^2 y}$  for some  $i \in [1, \ell-1]$ , then  $|h_{Q; e_i, \Delta_i}(Qx, Qy)| \geq \Lambda$ . Set  $h_i := h_{Q; e_i, \Delta_i}(Qx, Qy)$ . We have  $\min\{|h_1|, |h_{\ell-1}|, |h_1 - h_2|, \dots, |h_{\ell-2} - h_{\ell-1}|\} \geq \frac{\Lambda}{\ell}$ . Arguing exactly as in the proof of Lemma 11 it follows that the total contribution to (5.11) is  $\ll_{\ell, B} N^{-B}$  for every  $B > 0$ .

We infer that the set  $\tilde{\Omega}_{\mathbf{e}, \Delta, \mathbf{k}}(Q)$  can be replaced by the set  $\mathcal{D}_{\mathbf{A}, \mathbf{B}}$  defined in (1.3), where we took  $A_i = e_i k_i$ ,  $B_i = \Delta_i k_i$ ,  $\mathbf{A} = (A_1, \dots, A_{\ell-1})$ ,  $\mathbf{B} = (B_1, \dots, B_{\ell-1})$ . Note that  $\Psi_{F; \mathbf{e}, \Delta, \lambda} = \Psi_{F; \mathbf{A}, \mathbf{B}, \lambda}$ , as defined in (4.3). Using Proposition 6 we truncate the series in (5.11) at  $\max_i\{|A_i|, |B_i|\} \leq \Lambda$  and infer that

$$\begin{aligned} \mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) &= \frac{6Q^2}{\pi^2 N} \sum_{\substack{|A_i|, |B_i| \leq \Lambda, \forall i \\ k_i | (A_i, B_i), \forall i}} \mu(\mathbf{k}) \\ &\quad \times \iint_{\mathcal{D}_{\mathbf{A}, \mathbf{B}}} \Psi_{\hat{f}_\delta} \left( \frac{(N/Q^2)B_1}{y(A_1 y - B_1 x)}, \dots, \frac{(N/Q^2)B_{\ell-1}}{y(A_{\ell-1} y - B_{\ell-1} x)} \right) dx dy + O_{\ell, \varepsilon}(Q^{-\min\{\alpha_\ell, \delta_\ell\} + \varepsilon}). \end{aligned}$$

The convention here is that  $(A, 0) = A$  if  $A \in \mathbb{N}$ . Note that if  $B_i = 0$  for some  $i$  and  $\mathcal{D}_{\mathbf{A}, \mathbf{B}} \neq \emptyset$ , then  $A_i \geq 1$ .

Using again Proposition 6 we can truncate the sum at  $\max_i\{|A_i|, |B_i|\} \leq \Lambda^{\eta_\ell}$  for some small  $\eta = \eta_\ell > 0$  to be fixed, getting

$$\mathfrak{M}_{Q;\delta}^{(\Lambda^{\eta})}(\ell) = \mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) + O(\Lambda^{-\delta_\ell \eta_\ell}).$$

Using (5.1) we then replace  $\Psi_{\hat{f}_\delta}$  by  $\Psi_{\hat{\chi}_{[0,1]}} = \Psi_{\text{sinc}(\pi \cdot)}$  in the integral at the price of and error term which is  $\ll_\varepsilon \Lambda^{(2\ell-2)\eta_\ell + \varepsilon} \leq N^{-1+c+(2\ell-2)\eta_\ell d + \varepsilon}$ . This is acceptable so

long as we choose  $\boxed{0 < \eta_\ell < \frac{1-c}{(2\ell-2)d} < 1}$  and we infer

$$\begin{aligned} \mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) &= \frac{6Q^2}{\pi^2 N} \sum_{\substack{|A_i|, |B_i| \leq \Lambda^\eta, \forall i \\ k_i | (A_i, B_i), \forall i}} \mu(\mathbf{k}) \\ &\times \iint_{\mathcal{D}_{\mathbf{A}, \mathbf{B}}} \Psi_{\text{sinc}} \left( \frac{\pi(N/Q^2)B_1}{y(A_1 y - B_1 x)}, \dots, \frac{\pi(N/Q^2)B_{\ell-1}}{y(A_{\ell-1} y - B_{\ell-1} x)} \right) dx dy + O_\ell(Q^{-\theta_\ell}), \end{aligned}$$

with  $\boxed{\theta_\ell := \min\{\alpha_\ell, \delta_\ell \eta_\ell\} + \varepsilon < 1}$ . Finally, having replaced  $\Psi_{\widehat{f}_\delta}$  by  $\Psi_{\text{sinc}(\pi \cdot)}$ , we now extend the sum to all  $A_i \in \mathbb{Z}$  and  $B_i \in \mathbb{Z}$  with  $\mathcal{D}_{\mathbf{A}, \mathbf{B}} \neq \emptyset$  by again using Proposition 6. This contributes an error term of size  $O(\Lambda^{-\delta_\ell \eta_\ell})$ . After all these manipulations we conclude that

$$\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) = M_\ell \left( \frac{N}{Q^2} \right) + O_\ell(Q^{-\theta_\ell}), \quad (5.12)$$

where the quantity  $M_\ell(\alpha)$  defined as in (1.4) is absolutely convergent as a result of Proposition 6. Note that in fact the error term is a bit weaker, in the sense that if  $0 < \gamma_1 < \gamma_2$  are given, then the error above is  $\ll_{\ell, \gamma_1, \gamma_2} Q^{-\theta_\ell}$  whenever it is assumed that  $\gamma_1 Q^2 \leq N \leq \gamma_2 Q^2$ . This concludes the proof of part (i) in Theorem 1.

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